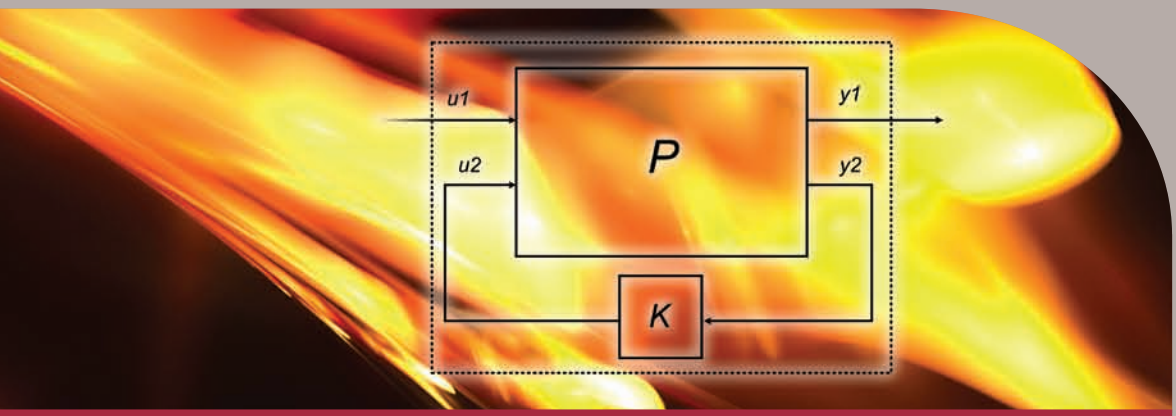
Control Engineering Series 72



Control Theory

A guided tour

3rd Edition

James Ron Leigh

IET CONTROL ENGINEERING SERIES 72

Control Theory

Other volumes in this series:

Volume 8 **A history of control engineering, 1800–1930** S. Bennett
Volume 18 **Applied control theory, 2nd edition** J.R. Leigh
Volume 20 **Design of modern control systems** D.J. Bell, P.A. Cook and N. Munro (Editors)
Volume 28 **Robots and automated manufacture** J. Billingsley (Editor)
Volume 33 **Temperature measurement and control** J.R. Leigh
Volume 34 **Singular perturbation methodology in control systems** D.S. Naidu
Volume 35 **Implementation of self-tuning controllers** K. Warwick (Editor)
Volume 37 **Industrial digital control systems, 2nd edition** K. Warwick and D. Rees (Editors)
Volume 39 **Continuous time controller design** R. Balasubramanian
Volume 40 **Deterministic control of uncertain systems** A.S.I. Zinober (Editor)
Volume 41 **Computer control of real-time processes** S. Bennett and G.S. Virk (Editors)
Volume 42 **Digital signal processing: principles, devices and applications** N.B. Jones and J.D.McK. Watson (Editors)
Volume 44 **Knowledge-based systems for industrial control** J. McGhee, M.J. Grimble and A. Mowforth (Editors)
Volume 47 **A history of control engineering, 1930–1956** S. Bennett
Volume 49 **Polynomial methods in optimal control and filtering** K.J. Hunt (Editor)
Volume 50 **Programming industrial control systems using IEC 1131-3** R.W. Lewis
Volume 51 **Advanced robotics and intelligent machines** J.O. Gray and D.G. Caldwell (Editors)
Volume 52 **Adaptive prediction and predictive control** P.P. Kanjilal
Volume 53 **Neural network applications in control** G.W. Irwin, K. Warwick and K.J. Hunt (Editors)
Volume 54 **Control engineering solutions: a practical approach** P. Albertos, R. Strietzel and N. Mort (Editors)
Volume 55 **Genetic algorithms in engineering systems** A.M.S. Zalzala and P.J. Fleming (Editors)
Volume 56 **Symbolic methods in control system analysis and design** N. Munro (Editor)
Volume 57 **Flight control systems** R.W. Pratt (Editor)
Volume 58 **Power-plant control and instrumentation** D. Lindsley
Volume 59 **Modelling control systems using IEC 61499** R. Lewis
Volume 60 **People in control: human factors in control room design** J. Noyes and M. Bransby (Editors)
Volume 61 **Nonlinear predictive control: theory and practice** B. Kouvaritakis and M. Cannon (Editors)
Volume 62 **Active sound and vibration control** M.O. Tokhi and S.M. Veres
Volume 63 **Stepping motors: a guide to theory and practice, 4th edition** P.P. Acarnley
Volume 64 **Control theory, 2nd edition** J.R. Leigh
Volume 65 **Modelling and parameter estimation of dynamic systems** J.R. Raol, G. Girija and J. Singh
Volume 66 **Variable structure systems: from principles to implementation** A. Sabanovic, L. Fridman and S. Spurgeon (Editors)
Volume 67 **Motion vision: design of compact motion sensing solution for autonomous systems** J. Kolodko and L. Vlacic
Volume 68 **Flexible robot manipulators: modelling, simulation and control** M.O. Tokhi and A.K.M. Azad (Editors)
Volume 69 **Advances in unmanned marine vehicles** G. Roberts and R. Sutton (Editors)
Volume 70 **Intelligent control systems using computational intelligence techniques** A. Ruano (Editor)
Volume 71 **Advances in cognitive systems** S. Nefti and J. Gray (Editors)
Volume 73 **Adaptive sampling with mobile WSN** K. Sreenath, M.F. Mysorewala, D.O. Popa and F.L. Lewis
Volume 74 **Eigenstructure control algorithms: applications to aircraft/rotorcraft handling qualities design** S. Srinathkumar
Volume 75 **Advanced control for constrained processes and systems** F. Garelli, R.J. Mantz and H. De Battista
Volume 76 **Developments in control theory towards glocal control** Li Qiu, Jie Chen, Tetsuya Iwasaki and Hisaya Fujioka (Editors)
Volume 78 **Frequency-domain control design for high-performance systems** John O'Brien

Control Theory

A guided tour

3rd Edition

James Ron Leigh

The Institution of Engineering and Technology

Published by The Institution of Engineering and Technology, London, United Kingdom

The Institution of Engineering and Technology is registered as a Charity in England & Wales (no. 211014) and Scotland (no. SC038698).

First published 2012

The Institution of Engineering and Technology
Michael Faraday House
Six Hills Way, Stevenage
Herts, SG1 2AY, United Kingdom

www.theiet.org

British Library Cataloguing in Publication Data
A catalogue record for this product is available from the British Library

ISBN 978-1-84919-227-9 (hardback)
ISBN 978-1-84919-228-6 (PDF)

Typeset in India by MPS Limited
Printed in the UK by CPI Group (UK) Ltd, Croydon, CR0 4YY

To James, Rachael, Harrison, Alice, Frances,
Molly, Elizabeth, Edward, Daniel, …

Contents

Acknowledgements **xix**

Foreword by Professor Karl Åström **xxi**

Introduction **xxiii**

The structure, content and purpose of the book xxiii

Some historical threads in the development of control systems technology xxv

1. Control concepts: a non-mathematical introduction **1**

1.1 General systems ideas 1

1.2 What is control theory? An initial discussion 2

1.3 What is automatic control? 6

1.4 Some examples of control systems 7

2. Control design ideas: a non-mathematical treatment **11**

2.1 Initial discussion 11

2.2 Question: can the best possible control actions be synthesised by some mechanism? 12

2.3 Requirements for an automatic control system 12

2.4 Automatic feedback control 14

2.5 Diagrams illustrating and amplifying some of the concepts described so far and showing relationships to a software engineering context 14

3. Synthesis of automatic feedback control loops: a more quantitative view **19**

3.1 Feedback loops: further discussion 19

3.2 What sorts of control laws are there? 21

3.3 How feedback control works: a practical view 22

3.4 General conditions for the success of feedback control strategies 27

3.5 Alternatives to feedback control 28

4. How the Laplace transform greatly simplifies system representation and manipulation **29**

4.1 Laplace transform techniques 29

4.2 Definition of the Laplace transform 29

4A *Convergence of the integral that defines the Laplace transform* 30

4B *Problems with* 0^- *and* 0^+ 31

4.3 Use of the Laplace transform in control theory 31

4.4 The concept of transfer function 32
4.5 System simplification through block manipulation 32
4.6 How a transfer function can be obtained from a differential equation 33
4.7 Poles and zeros of a transfer function 33
4.8 Understanding system behaviour from a knowledge of pole and zero locations in the complex plane 34
4.8.1 Meaning of pole locations 34
4.9 Pole placement: synthesis of a controller to place the closed loop poles in desirable positions 35
4.10 Moving the poles of a closed loop system to desirable locations: the root locus technique 38
4.11 Obtaining the transfer function of a process from either a frequency response curve or a transient response curve 39
4.11.1 Obtaining a transfer function from a given frequency response curve 39
4.11.2 Obtaining a transfer function from a transient response curve 40
4C *Convolution: what it is* 40
4.12 Determination of transfer functions by cross-correlation 41
4D *Calculation of resonant frequencies from the pole–zero diagram* 42
4E *Derivation of a formula for damped natural frequency* 43
4F *The root locus of a system with open loop poles and zeros located as in Figure 4.19 will include a circle centred on the zero* 45

5. Frequency response methods **47**
5.1 Introduction 47
5.2 Design using frequency response methods: initial explanation 47
5.3 Frequency response of a linear system 47
5.4 The Bode diagram 48
5.5 Frequency response and stability: an important idea 48
5.6 Simple example of the use of the foregoing idea in feedback loop design 50
5.7 Practical point: the need for stability margins 50
5.8 General idea of control design using frequency response methods 51
5.9 Obtaining the frequency response of a system experimentally 51
5.9.1 Obtaining the frequency response of a system experimentally: some practical difficulties 52
5.10 Design based on knowledge of the response of a system to a unit step input 53
5.11 How frequency response is obtained by calculation from a differential equation 54
5.12 Frequency response testing can give a good estimate of a system's transfer function 56

5.13 Frequency response of a second-order system 57
5A *The frequency response of a system with poles and/or zeros near to the imaginary axis* 58
5B *Some interesting and useful ideas that were originated by Bode* 59
5.14 Nyquist diagram and Nichols chart 60

6. Mathematical modelling 63
6.1 Approaches to mathematical modelling 63
6.2 Methods for the development of mathematical models 64
6.3 Modelling a system that exists, based on data obtained by experimentation 64
6.4 Construction of models from theoretical considerations 66
6.5 Methods/approaches/techniques for parameter estimation 66
6.6 Why modelling is difficult: an important discussion 68
6.7 Fixing of parameters 69
6.8 Parameter estimation 69
6.9 Regression analysis 69
6.10 Analysis of residuals 70
6A *Doubt and certainty* 71
6B *Anticipatory systems* 72
6C *Chaos* 74
6D *Mathematical modelling: some philosophical comments* 76
6E *A still relevant illustration of the difficulty of mathematical modelling: the long march towards developing a quantitative understanding of the humble water wheel, 1590–1841* 79
6F *A postscript on the effectiveness of thought experiments* 82
6G *Experimentation on plants to assist in model development: the tests that you need may not be in the textbook!* 83
6H *Dimensional analysis* 85
6I *Distributed systems: the cosy idealisation that a system can be characterised by its behaviour at a point in space* 87
6J *When can the behaviour in a region be well approximated at a point?* 89
6K *Oscillating and osculating approximation of curves* 90

7. Non-linear systems 93
7.1 What is meant by non-linearity 93
7.2 Approaches to the analysis of non-linear systems 95
7.2.1 Lyapunov's second or direct method 95
7.2.2 Lyapunov's first method 95
7.2.3 Describing function method (described later in this chapter) 95
7.2.4 Sector bound methods 96
7.3 The describing function method for analysis of control loops containing non-linearities 97

7.4 Linear second-order systems in the state plane 99
7.5 Non-linear second-order systems in the state plane 100
7.6 Process non-linearity: large signal problems 101
7.7 Process non-linearity: small signal problems 102
7.8 Linearisation 103
7.8.1 The motivation for linearisation 103
7.9 What is linearisation? 104
7.9.1 An initial trivial example 104
7.9.2 Comments 105
7.10 Linearisation about a nominal trajectory: illustration 106
7.11 The derivative as best linear approximation 107
7A *The inverse function theorem* 107
7B *The concept of transversality* 108
7C *The forced pendulum: a simple example of chaotic behaviour* 109

8. Limits to performance 111
8.1 Stability: initial discussion 111
8A *Stability theory: a long-term thread that binds* 112
8.2 Stability for control systems: how it is quantified 113
8B *The ingenious method of Hurwitz* 115
8.3 Linear system stability tests 116
8.4 Stability margin 116
8.5 Stability tests for non-linear systems 117
8.6 Local and global stability 118
8.7 Lyapunov's second (direct) method for stability determination 119
8C *Geometric interpretation of Lyapunov's second method* 120
8.8 What sets the limits on the control performance? 121
8.9 How robust against changes in the process is a moderately ambitious control loop? 124
8.10 Limits and constraints on synthesis: summary of points 126
8.11 Systems that are difficult to control: unstable systems 126
8.11.1 Cancellation of an unstable pole by a matching zero in the controller 127
8.11.2 Shifting an unstable pole by feedback 128
8D *Control of quiet vertical standing in humans* 129
8E *Understanding, with the help of the full Nyquist stability criterion (Nyquist, 1932), how simple feedback control can stabilise an unstable process* 130
8F *Examples of systems that are deliberately designed to be unstable* 133
8.12 Systems that are difficult to control: non-minimum phase systems 135
8.12.1 The effect of system zeros on the step response of a system 136

8.13 Process dead time: a difficult dynamic element in the control loop 139
8.13.1 How to control processes that have significant dead time 140
8.14 Some interesting theoretical limitations on performance 141
8.14.1 Sensitivity functions and their interrelation 141
8G *Motivation for the name: non-minimum phase systems* 141
8.14.2 Integral constraints in the time domain 143
8.14.3 Design constraints caused by Bode's theorem 144
8H *Mapping of complex functions: a few points that underlie classical control theory* 145
8I *Derivatives of a complex function* G(s) 147
8J *Singularities of a complex function* G(s) 147

9. Some practical aspects of control design, implementation and justification 149
9.1 The variables that really matter may be difficult to define, measure or control 149
9.2 How efficient is it to control a process by a controller that consists only of a high gain of value *C* that, in the limit, becomes a relay controller? 149
9.2.1 Rudimentary on–off control 149
9.2.2 Introduction to variable structure systems and sliding mode control 151
9.3 An off-the-shelf approach to the control of an unknown process 155
9.3.1 The three-term controller 155
9.3.2 Illustration of the value of an integral term in removing any constant error 156
9.3.3 Illustration of the value of a derivative term to control the degree of damping 156
9.3.4 How can the three coefficients of a three-term controller be chosen quickly in practice? 158
9A *How to learn something from the first part of a step response* 163
9B *New York to San Francisco telephony: an early illustration of the spectacular success of feedback in achieving high-fidelity amplifications of signals* 165
9.4 Control systems for batch process 166
9C *The idea of a probing controller* 169
9.5 Input shaping 169
9.6 Gain scheduling (to allow a control system to operate successfully when the process to be controlled changes its characteristics over so wide a range that no constant controller can be found that performs adequately) 172
9.6.1 Gain scheduling: traditional ad hoc and modern linear parameter varying (LPV) based 173

9.6.2 LPV gain scheduling as a step forward from traditional gain scheduling 173
9.6.3 Outline of the LPV approach to gain scheduling 175
9.7 Converting a user's requirements into a control specification 175
9.8 Methodologies for deciding the scope and attributes of automatic control schemes for industrial application (in particular, methodologies for economic justification of investment in automation) 177
9.8.1 Methodologies and illustrations 177

10. Discrete time and digital control systems 185
10.1 Introduction 185
10.2 Computers as system components: devices that can change their state only at discrete times 185
10A *A simple and informative laboratory experiment* 187
10.3 Discrete time algorithms 187
10.4 Approaches to algorithm design 188
10.4.1 Direct controller synthesis 188
10B *A clever manipulation: how the digital to analogue convertor (zero-order hold) is transferred for calculation purposes to become part of the process to be controlled* 190
10.4.2 Gain plus compensation approach 190
10C *Takahashi's algorithm* 191
10.5 Overview: concluding comments, guidelines for algorithm choice and some comments on procedure 193
10D *Some difficulties in moving from differential equations to approximating difference equations* 193
10E *Discretisation* 194
10F *A simple matter of quadratic behaviour* 194
10G *Continuous is not the limit of discrete as* $T \to 0$ 197
10H *Non-uniqueness of inverse Z transforms* 197
10I *The stability of a system can usually be considered independently of the nature of the inputs that it receives, but here is a counter-example* 198

11. Multivariable linear systems and the state space approach 199
11.1 Introduction 199
11.1.1 State space representations 199
11.1.2 The state vector 199
11.2 The concept of state 201
11.3 Alternative system descriptions 202
11.4 The mapping representation of Σ 203
11.5 The modelling of continuous time systems through state space equations 205

11.5.1 A general non-linear model 205
11.5.2 Linearisation of the model 205
11.6 Calculation of time solutions using the transition matrix 207
11.6.1 The time-invariant case 207
11.6.2 The time-varying case 211
11.6.3 The periodically time-varying case 214
11.7 Properties of the transition matrix 215
11.8 Relation between the transfer-matrix description and the vector–matrix description 215
11.9 Equivalent systems 216
11.10 System realisation 216
11.10.1 Existence 217
11.10.2 Uniqueness 217
11.11 Stability 217
11.11.1 Stability tests for continuous time systems 217
11.12 Reachability, controllability, observability and reconstructibility for continuous time systems 218
11.12.1 Controllability and observability tests for continuous time, time-invariant systems by the matrices Q_c, Q_o 218
11.12.2 Controllability and observability tests for continuous time, time invariant systems by the diagonalisation method 219
11.13 The unforced state equation in discrete time 219
11.13.1 Existence and uniqueness of solution 219
11.13.2 The transition matrix for the discrete time equation 220
11.14 Solution of the discrete time state equation with forcing 221
11.15 Obtaining the $\mathscr{Z}$ transform equivalent of the state equation 221
11.16 Stability tests 222
11.17 Reachability, controllability, observability and reconstructibility for discrete time systems 222
11.18 Canonical state space representations 224
11.18.1 Introduction 224
11.18.2 The reachability canonical form 226
11.18.3 The controllability canonical form (phase-variable form) 228
11.18.4 The observability canonical form 231
11.18.5 The reconstructibility canonical form 232
11.18.6 State equations for multi-input, multi-output processes 233
11.18.7 The Jordan canonical form 234
11.19 The state-variable approach to control system design 236
11.20 Design to achieve a specified closed loop characteristic equation 237
11.20.1 Control design based on state-variable feedback 237
11.20.2 Modal control by state feedback 239
11A *It seems remarkable that an oscillatory solution can be generated by repeated multiplication by a constant matrix (Figure 11.15)* 245

11B *Conservation of dimension under linear transformations* 245
11C *A deeper look at controllability and observability* 247

12. Links between state space and classical viewpoints **249**
12.1 Introduction 249
12.2 A state space view of cascade control 250
12.2.1 Establishing the state space equations by inspection from the block diagram 251
12.2.2 Bringing the inner loop into the model 253
12.2.3 Looking at the performance of the jacketed reactor under control, first with the single loop and then, for comparison purposes, with the inner loop operational 253
12.3 An inverse Nyquist view of the entries in the *A* matrix of a system representation 254
12.4 Illustration of modes and modal analysis 256
12A *An interesting approach to modal analysis for a third order system with two complex eigenvalues* 261
12.5 Moving between different system representations: the relationship between state space and transfer function representations 264
12.5.1 Poles and zeros of state space systems 265
12.6 The relative gain array (RGA) and singular value decomposition: two useful tools to help in the task of controlling a multivariable process using a set of individual SISO control loops 266
12.6.1 The RGA: in its simplest form, a forecaster of steady state interaction 266
12.6.2 Illustrative example 266
12.6.3 Singular value decomposition 268

13. Optimisation **271**
13.1 Initial discussion 271
13.2 Optimisation: a few ideas that can form building blocks 271
13.2.1 Discussion 277
13.3 Time-optimal control 278
13A *Time-optimal control: a geometric view* 280
13B *The following geometric argument can form a basis for the development of algorithms or for the proof of the Pontryagin maximum principle* 283
13C *Construction of time-optimal controls* 284
13D *The (matrix) Riccati equation: some history* 285
13.4 Linear quadratic optimisation 286
13.4.1 LQR problems with infinite time horizon 286
13.4.2 LQR problems with finite time horizon 292

14. State estimation: observers and the Kalman filter and prediction 299
14.1 Introduction 299
14.2 The separation principle 299
14.3 State estimation: what it attempts to do 300
14.4 How a state estimator works: the Kalman filter 300
14.5 The Kalman filter: more detail 301
14.6 Obtaining the optimal gain matrix 303
14.7 Prerequisites for successful application of the Kalman filter in the form shown in Figure 14.4 304
14.8 Discussion of practical points arising 305
14.8.1 Use of the innovation sequence to modify R and Q 306
14A *A Matlab demonstration of the Kalman filter at work, estimating a true system output from the noise-corrupted output* 306
14.9 Prediction and predictive control 307
14B *Model Predictive Control (MPC), also known as predictive-iterative control* 308

15. An introduction to robust control design using H_∞ and related methods 309
15.1 Motivation and introduction 309
15.2 Hardy spaces (H_p, $p > 0$ spaces) and their relevance to control problems 310
15.2.1 The suffix p 310
15.2.2 Elementary illustration: the effect of choice of p on the nature of the norm 310
15.2.3 Non-elementary aside 311
15.3 A simple view of H_∞ control loop design 312
15.3.1 Guaranteed stability of a feedback loop 312
15.3.2 Robust stability of a closed loop 313
15.4 Total internal stability and design for disturbance rejection 314
15.4.1 Setting the scene 314
15.4.2 Robust stability 315
15.4.3 Disturbance rejection 317
15.5 Robust control design using a mixed sensitivity H_∞ loop shaping approach: worked example 318
15.6 Specification of the ΔG envelope 322
15.7 Deriving H_∞ norms from dynamic process models 322
15.7.1 Singular values and eigenvalues 322
15.7.2 Eigenvalues of a rectangular matrix A 324
15.7.3 Singular values and their relation to eigenvalues 325
15.7.4 Relations between frequency and time domains 325
15.8 A promising visualisable design tool that works within the H_∞ frame: the ν gap metric 326
15.8.1 Introduction 326
15.8.2 Simple illustration of the use of the ν gap metric 326
15.8.3 More about the two metrics $\delta\nu$ and $b_{G,D}$ 327

15.8.4 The insight provided by the ν gap metric 328
15.8.5 Taking into account the weights shown in Figure 15.16 to allow loop shaping 329
15.8.6 A discussion on the two metrics $\delta\nu$ and $b_{G,D}$ 329
15.9 Using LMI methods in control systems analysis and design 330
15.9.1 Definition of an LMI 332
15.9.2 Solving an LMI: the feasibility stage: worked example 332
15.9.3 LMI applications to control: simple examples 333
15.9.4 Motivating example: robust pole placement using LMIs 334
15.10 An outline of how H_∞ design works and how its practicality can be usefully extended through μ synthesis 335
15.10.1 The μ-synthesis method 336
15.11 Robustness or adaptivity? 337
15A *A hierarchy of spaces* 337
15B *Model reduction to avoid overcomplexity in synthesised controllers* 340

16. A miscellany of control techniques 343
16.1 Introduction 343
16.2 Artificial neural networks (ANN) 344
16.2.1 Motivation 344
16.2.2 A basic building block: the neuron 345
16.2.3 Simple properties of a neuron demonstrated in the two-dimensional real plane 345
16.2.4 Multilayer networks 347
16.2.5 Neural network training 347
16.2.6 Neural network architectures to represent dynamic processes 349
16.2.7 Using neural net-based self-organising maps for data reduction and clustering 352
16.2.8 Upcoming rivals to neural networks? Support vector machines and adaptive logic networks 352
16.2.9 Neural nets: summary 355
16.3 Fuzzy set theory and fuzzy logic 353
16.3.1 Introduction and motivation 353
16.3.2 Some characteristics of fuzzy logic 355
16.3.3 References: early pioneering work 356
16.4 Genetic algorithms 356
16.4.1 Basic ideas 356
16.4.2 Artificial GAs 356
16.4.3 GAs as design tools 358
16.4.4 GA summary 358
16.4.5 References 359
16.4.6 Rivals to GAs? Autonomous agents and swarms 360

16.5 Controller switching, gain scheduling and adaptivity 360
16.5.1 Discussion 360
16.5.2 Controller switching 360
16.5.3 Gain scheduling 361
16.5.4 Adaptivity and self-tuning controllers 361
16.6 Learning systems (systems that learn) with or without supervision 361
16.6.1 Basic ideas 361
16.6.2 Learning versus adaptivity 362
16.6.3 Structural characteristics of an abstract learning system 362
16.7 Intelligent systems 364
16.7.1 The properties that an intelligent system ought to possess 364
16.8 Where next for AI techniques? 365
16A *The idea of a probing controller* 365

17. Review: the development of the control systems discipline and the mathematical roots of control systems theory 367
17.1 A rapid review of how control technology developed 367
17.2 The development of the control systems discipline: a structure 370
17.3 The mathematical roots of control theory 371

18. Resources, references and further reading 373
18.1 General remarks on the control literature and on the following references and recommended further reading 373
18.2 Finding books, papers, theses 374
18.2.1 Library books 374
18.3 Control-oriented software 374
18.4 Mainstream control literature 375
18.5 Older mainstream control books 375
18.6 Methodologies for economic justification of investment in automation 375
18.7 State estimation 376
18.8 Sliding mode control 376
18.9 Optimisation 376
18.10 Robust control 376
18.11 Neural networks and support vector methods 377
18.12 Fuzzy logic and fuzzy control 377
18.13 Genetic algorithm, genetic programming and other parallel evolutionary search methods 377
18.14 Intelligent and learning systems 378
18.15 Adaptive and model-based control 378
18.16 Stochastic aspects of control 378
18.17 Some other control topics 379

18.18 General mathematics references 379
18.19 Ordinary differential equations 379
18.20 Differential topology/differential geometry/differential algebra 380
18.21 Theory of equations 381
18.22 Operator theory and functional analysis applied to linear control 381
18.23 Books of historical interest 382
18.23.1 History of the Riccati equation 382
18.24 Miscellany 382
18.25 Useful tables 383
18.26 Alphabetical list of references and suggestions for further reading 383

Appendix A: Case histories **407**
Introduction 407
A1 Control of product thickness in a strip rolling mill: from a control point of view this is predominantly a dead time problem 407
Causes of thickness deviations 410
A2 The cut-up problem 411
A3 Control of the pressure inside small fuel-fired furnaces 414
A4 Batch control: a brief case history of one aspect: Oxygen Steelmaking 417
A5 A note on the introduction of novel measurement sensors into control systems 419

Notation **421**

Afterword **423**

Index **425**

Acknowledgements

I am delighted to thank the following professional colleagues for their assistance or for their willingness to allow me to use extracts from their publications:

Professor Derek Atherton of Sussex University for his many helpful suggestions during the planning of this book and its predecessors.

Professor Guy Beale of George Mason University for permission to make use of his LQR examples in Chapter 13.

Dr Victor M. Becerra, Reader in Cybernetics, University of Reading, for providing his loop shaping example in Chapter 15.

Professor Mogens Blanke, Technical University, Denmark, for advice and suggestions about the dynamics of ship steering.

Mr John Grafton, Commissioning Editor at Dover Publications, East Street, New York, for kindly allowing me to quote extensively from my Dover book, Leigh (2006).

Professor Karl Åström of Lund University for permission to reproduce Figure 9.3 from the book Åström and Hägglund (2006).

Dr Peter Turner, School of Engineering and Design, Brunel University, for working with me on the input-shaping demonstration problem and supplying the graph on that topic.

Dr Glen Vinnicombe of Cambridge University for permission to adapt examples from his book (Vinnicombe, 2002), for use in Chapter 15 on the ν gap metric.

Professor Stanislav Zak at Purdue University for allowing me to use Sliding Mode tutorial examples from his book Zak (2003).

The editorial staff at the Institution of Engineering and Technology for their helpfulness throughout.

The production staff at MPS Limited for their professionalism in helping to turn my manuscript into a book.

Foreword

This is a very unusual book on control reflecting the author's personal perspective that it is essential for control engineers to master the mathematical foundations as well as the applications and the history of the field. The book covers a very wide range from intuitive discussions of control to sophisticated technical details. It is sprinkled with historical notes and philosophical reflections; there is an initial emphasis on concepts rather than on practical detail.

The style is similar to the book *Mathematical Modelling Techniques* by Rutherford Aris (1978), although that book deals with a much more limited domain.

The first edition of the book was published in 1992, the organisation of the present third edition remains essentially the same. Nine chapters are basically as before, with minor edits, additions and reorganisations. The remainder have been updated, in particular to include Matlab and Scilab examples that illustrate how those tools can be used to solve realistic industrial problems.

The early chapters introduce concepts and ideas without mathematics; they are followed by several chapters dealing with classical control, Laplace transforms, frequency response, modelling and non-linear systems. Then follow updated practically oriented chapters on *limits to performance* and *real-world implementation*, which reflect the author's industrial control design experience. The main innovation of this new edition is the introduction of the two linked chapters *Multivariable linear systems and the state space approach* and *Links between state space and classical viewpoints*. The first takes a formal mathematical viewpoint, whereas the second is an informal explanatory chapter. These two chapters are, in a sense, the core of the book, showing the insight that can be gained by combining classical transfer function concepts with state space theory. Then follow chapters on Kalman filtering, optimisation, H_∞ control, soft-computing/artificial intelligence and the history of the mathematical roots that underpin the control systems subject. This edition also contains new topics such as input shaping, sliding-mode control, linear parameter varying (LPV) gain scheduling and linear matrix inequality (LMI) methods for solving optimisation problems.

There is an extensive set of references to suit a wide range of readers, including the historically interested. The book concludes with an appendix consisting of five real case histories drawn from the author's engineering experiences to illustrate the importance of the industrial context.

It is amazing to cover such a wide range in a single book but Leigh pulls it off by selecting a few key facts, omitting details and providing lots of references.

The use of control is spreading more and more, today you find control systems everywhere, in our homes, our cars, in factories and in scientific instruments. Control is also an essential element of all life processes. The need to know about control is therefore spreading. The first two editions of this unusual book reached a wide audience, I hope that this third edition will follow in their footsteps. Personally I enjoyed reading it, even if I do not always agree with the author's views.

Karl Johan Åström, Professor Emeritus, Lund University

Introduction

The structure, content and purpose of the book

The book is structured around a number of concepts that are central to control theory. They are presented with a minimum of detail 'only the raisins from the cake' but are frequently made clear by examples.

The first three chapters contain no mathematics at all, since it is intended that these chapters will form a useful introduction to the control subject for a wide class of readers. These early chapters largely answer the questions:

- What is control theory?
- What are the main ideas?
- What are the features that make the subject so fascinating, absorbing and universally useful?

The features of the book may be summarised:

- Emphasis on concepts.
- Follow up for the reader by reference links from the text to a very wide bibliography.
- First three chapters entirely non-mathematical: the next three chapters are gently introductory.

Attempts to apply theoretical control ideas to industrial situations are frequently constrained or prevented by the realities of the world context. Two chapters describe the generic aspects of applications that are most likely to limit what can be achieved in real life.

Appendix A gives a number of Industrial project histories, all based on the author's own experiences. They are intended to demonstrate the often overriding importance of individual process idiosyncrasies , compared with the control-theoretic aspects.

The large number of interludes is intended to stimulate interest. Appearing in a distinctive typescript, they may be omitted without detriment in a first reading of the mainstream text.

There is an extensive annotated bibliography.

The intended readership for the book is:

- Students working at any level on control engineering. Despite the multiplicity of available control books at all levels, students still struggle to

understand basic concepts. This book is intended as their companion and friend.

- Students of science, computing, mathematics and management. The book will supply these students with the main concepts of control, thus supporting the auxiliary control courses that are attended by these students.
- Industrialists, managers and professionals in a wide variety of fields. A large number of professionals from a wide variety of fields wish to understand the fundamentals and the potential of control, to an extent that will demystify the subject and that will allow them more effectively to assess the benefits of control to their particular areas.
- Engineers already familiar with control. They will hopefully find the book enjoyable, paralleling the enjoyment that I have obtained from writing it.

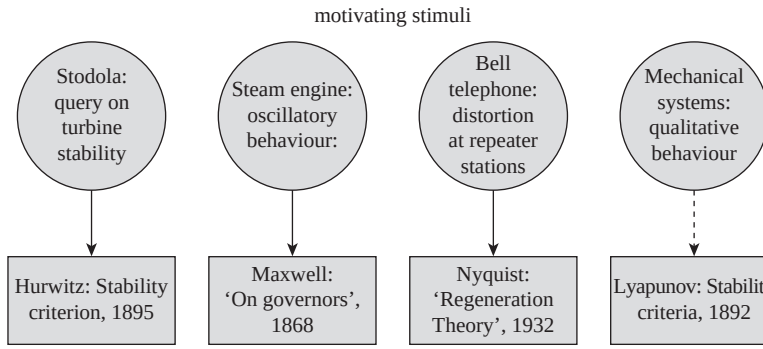
Every worthwhile discipline has a strong structure and underlying principles and is possessed of a continuous striving towards improved coherence so that what, at first sight, appeared to be isolated phenomena take their place in the structure in a consistent way. Thus, the science of physics has been brought, by generations of dedicated development, to its present well-unified state.

Here, we are concerned with the structure, principles and context of control theory.

Control theory is a very powerful body of knowledge indeed. It allows the synthesis of systems having specified characteristics. It can model and include within its control loops any complex object (for instance, an aircraft) that needs to be so included. It can produce adaptive solutions that change automatically as circumstances change. It can combine with pattern recognition, with expert systems and with artificial intelligence (AI) in general. It makes use of computer power to identify problems, to solve problems, to validate solutions and to implement final solutions. Control has an impressive track record of successful applications across aircraft, ships, satellite and missile guidance, process industries (chemicals, oil, steel, cement etc.), pharmaceuticals, domestic and computer goods (automatic cameras etc.), public utilities (e.g. all aspects of electrical generation and supply), automatic assembly, robotics, prosthetics and increasingly it lends its basic ideas to other disciplines.

Control theory is built up around a few very simple ideas: such ideas as feedback loop and stability. The writing of this book has been motivated by a belief that it is absolutely vital and worthwhile to obtain a robust understanding of these few simple ideas and not allow them to be submerged below a cloud of techniques or numerical detail.

Some historical threads in the development of control systems technology



The discipline of control systems has strong historical roots that underpin its present-day strength. Many rival schemas might be put forward to identify the main such roots and here you have the author's choice. Interestingly (and perhaps inevitably) the four stimuli were all triggered by stability questions arising from particular applications.

The Czechoslovak engineer Stodola, constructing an early hydroelectric station at Davos in Switzerland had produced 11 linked equations describing the dynamics of the plant but could not solve them to check the stability of the proposed station. He therefore discussed the situation with his ETH[1] Zurich colleague Adolf Hurwitz, who, in response, produced his Hurwitz criterion [Hurwitz (1895)] that provided sufficient stability information from the coefficients of the equations without any need to solve them.

The Watt governor, an ingenious device that automatically controlled the speed of steam engines, was at one time of great importance, but it suffered from oscillatory behaviour, particularly as the steam engines became larger and more powerful. A proper analysis of the dynamics of the governor had to wait until Maxwell (1868) whose masterly analysis can be read in his paper 'On Governors'. Norbert Wiener wrote[2] 'we wish to recognise that the first significant paper on feedback mechanisms is an article on governors, which was published by James Clerk Maxwell in 1868'.

Telegraphic and telephonic communication may seem an unlikely application area to be interested in stability. However, it was the need for low distortion repeating amplifiers that could be used many times sequentially in a USA coast to coast communication link that drove the research that led to the

[1]Eidgenössische Technische Hochschule.
[2]Norbert Wiener in *Cybernetics: or Control and Communication in the Animal and the Machine* (Cambridge, MA, MIT Press, 1948), pp. 11–12.

quantitative understanding of feedback loops as described by Nyquist and Black, both from Bell Labs.

Nyquist's contribution (Nyquist 1932) is widely recognised but the work of Harold Black (Black 1934) has some claims to priority. To quote Black on this: 'it was on August 2, 1927 that the concept of the negative feedback amplifier came to me in a flash while I was crossing the Hudson River on the Lockawanna Ferry, on my way to work.'

Both the paper by Nyquist and the one by Black can be seen in the 2001 compilation, (Banjar, 2001). See also Harold S. Black, 'Inventing the negative feedback amplifier', *IEEE Spectrum*, vol. 14, pp. 54–60, Dec. 1977.

Chapter 1

Control concepts: a non-mathematical introduction

1.1 General systems ideas

The objects under study in control theory are systems. A system is any set of elements connected together by information links within some delineated system boundaries. Referring to Figure 1.1, note that the system boundary is not a physical boundary but rather a convenient fictional device. Note also how information links may pass through the system boundary.

Since control theory deals with structural properties, it requires system representations that have been stripped of all details, until the main property that remains is that of connectedness. (The masterly map of the London Underground system is an everyday example of how useful a representation can be when it has been stripped of all properties except that of connectedness.)

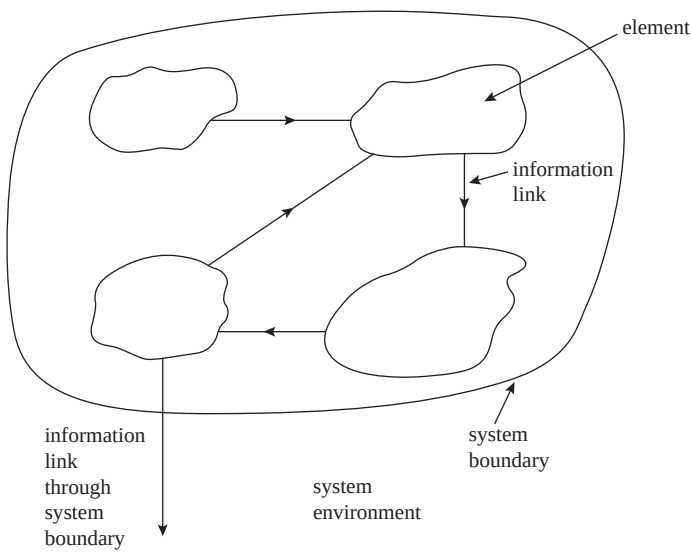


Figure 1.1 The structure of a system

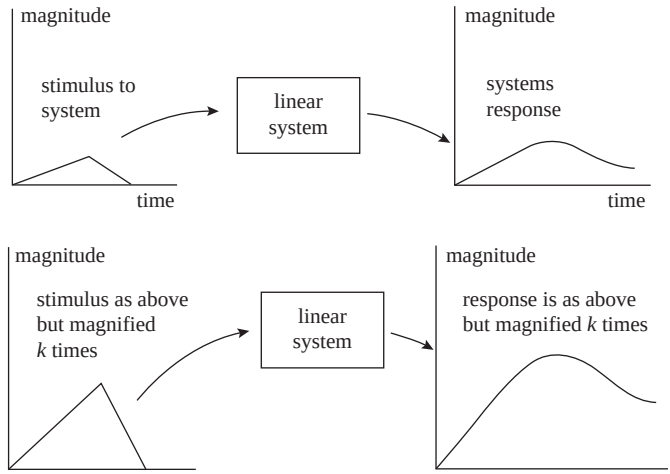


Figure 1.2 Linear system characteristics

Connectedness is a concept from topology. Topology, the discipline that studies the underlying structure of mathematics, offers fascinating reading to aspiring systems theorists. Recommended reading is given in the Bibliography. Clearly, a system is a very general concept; control theory is most interested in certain classes of system, and to make progress, we delineate the classes. First, it is interested in dynamic systems – these are systems whose behaviour over a time period is of interest. Thus, if a system were concerned with population aspects, a similar dynamic system would be concerned with population growth.

Second, it is most interested in and most powerful when dealing with linear systems. A linear system is characterised by the property shown in Figure 1.2. The upper part of the figure shows a system's response to some arbitrary stimulus. The lower part shows how, in the presence of linearity, the response to a scaled-up version of the stimulus is simply a scaled-up version of the previous response, with proportionality being preserved.

Finally, it is interested in feedback systems – these are systems where information flows in one or more loops, so that part of the information entering an element may be information that previously left that element (Figure 1.3).

Systems are often visualised in the form of block diagrams, illustrating the main functions, their supposed interconnection and (possibly) their interconnection to the environment of the system. Thus, a simple representation of the human temperature regulation system might be as shown in Figure 1.4.

1.2 What is control theory? An initial discussion

Many areas of study are fortunate in that their titles trigger an immediate image of their scope and content. For instance, the names 'human anatomy', 'veterinary

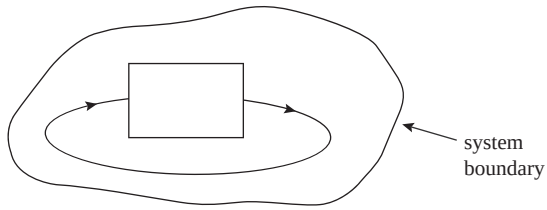


Figure 1.3 A simple feedback system

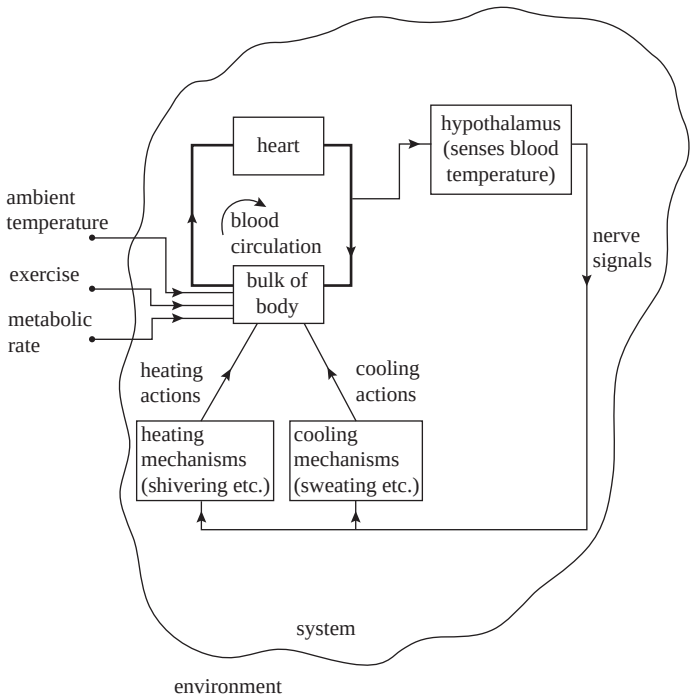


Figure 1.4 A simple representation of the human temperature regulation system

medicine', 'aeronautical engineering' and 'ancient history' all conjure up coherent visions of well-defined subjects. This is not so for control theory although almost everyone is interested in control in the sense of being able to achieve defined objectives within some time frame. Rather specific examples occur in the named professions of 'financial controller' and 'production controller'.

Control theory applies to everyday situations, as in the examples given above, just as well as it applies to the more exotic task of manoeuvring space vehicles. In fact, the concepts of control theory are simple and application independent. The universality of control theory means that it is best considered as applied to an abstract situation that contains only the topological core possessed by all situations that need to be controlled. Such an abstract situation is called a system.

The argument is that if we know how to control a highly general situation called a system then we shall be able to control any and every particular situation. This is the viewpoint of control theory and it is this viewpoint that gives it its extraordinary power.

Thus, any situation, delineated from its environment for study, is called a system. When control theory wishes to study temperature regulation in the human body, it concerns itself with a system involving blood circulation, heat generation and heat loss mechanisms and decision making by the brain. Systems can usefully be defined in almost any discipline – they are not confined to science or engineering.

Control theory concerns itself with means by which to alter the future behaviour of systems. For control theory to be successfully applied, the following needs to be available:

(i) A purpose or objective that is linked with the future state of the system. (Clearly the past cannot be influenced nor can the present, since no response can take place in any system in zero time.) The objective of any control system in every case is connected with the performance of the system over some period of time – the accountant and the industrial manager want to see long periods of smooth and profitable operation. Sometimes this leads to conflicting requirements, in the sense that short-term objectives are frequently in direct opposition to long-term objectives. In general terms, this objective can be considered to be the desired behaviour of the system.

(ii) A set of possible actions that offers an element of choice. (If no variation of actions is possible, control cannot be exercised and the system will follow a course that cannot be modified.)

(iii) Unless a trial-and-error strategy is to be adopted, some means of choosing the correct actions (ii) that will result in the desired behaviour (i) being produced.

In general terms, this requirement is met by a model capable of predicting the effect of control actions on the system state. Such a model may be implicit and not even recognised as a model or it may consist of a large and complex set of equations.

For the accountant, the model is a balance sheet together with inherited wisdom. For the military commander, the model is a map of local terrain and a knowledge of the types and deployments of men and equipment. For the control of quantities that can be measured by sensors, mathematical models in the form of stored curves or sets of equations will usually be used.

We see then that to achieve successful control we must have a defined objective and be able to predict adequately, over some sufficient time scale, all the outcomes of all the actions that are open to us. Figure 1.5 summarises the three requirements needed for successful control.

A major problem in control using a long-term horizon is uncertainty of the long-term accuracy of models, compounded by the likelihood of unforeseen events. That is to say, the possibility must be faced that, once uncertainty rises above a particular level, no meaningful control can be implemented and that policies that look ahead to anticipate future contingencies may call for immediate sacrifices that will never be repaid by the creation of more favourable future environments.

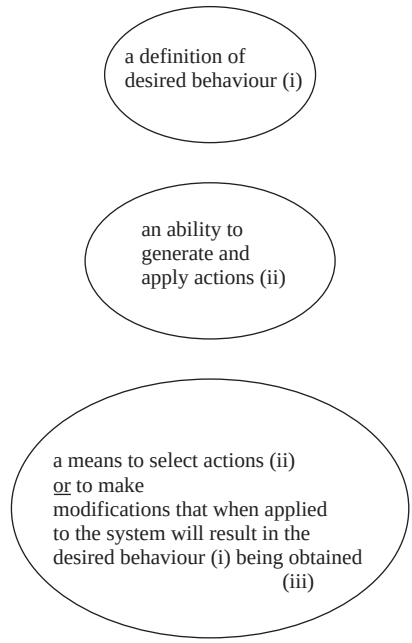


Figure 1.5 The three elements needed for successful control design

Feedback control, in which an error initiates corrective action, can be used only where corrective actions take effect relatively quickly. It is clearly unsatisfactory to wait until electricity demand exceeds the maximum possible supply level before starting to build a new power station and there is clearly a need for prediction. On the other hand, it is usually perfectly feasible to control the speed of a motor by an error-driven feedback correction.

None of the processes that we are called upon to control can be made to change its state instantaneously. This is because all processes have the equivalent of inertia. Suppose that we have the task of moving a large spherical boulder from A to B by brute force (Figure 1.6). Clearly, considerable initial effort must be expended to get the boulder rolling and a similar effort must be expended to bring it to rest. In the case illustrated, it will be all too easy to overshoot the target or to spend too long

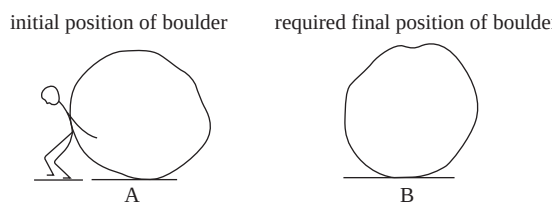


Figure 1.6 The problem of moving the boulder

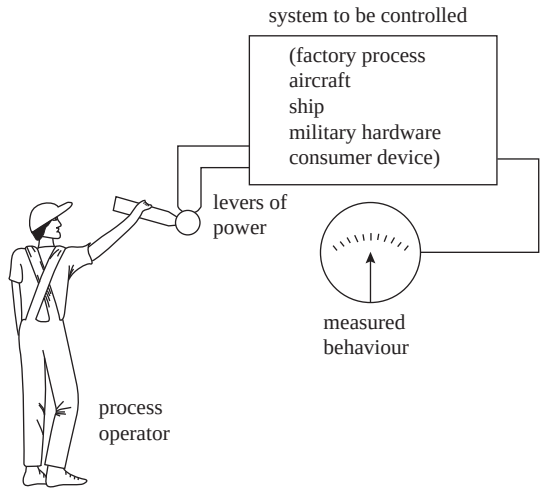


Figure 1.7 A manually controlled process

arriving there if any miscalculation is made. The difficulty of achieving control in this situation is entirely typical and occurs because of the energy that needs to be stored in and then removed from the boulder to allow the task to be achieved. Only when we possess a prior quantitative knowledge of the energy storage mechanism can we hope to achieve fast and accurate control.

A system with internal energy storage is called a dynamic system. Thus, we can see that one of our chief problems is to synthesise actions that, when applied to a dynamic system, will produce the response that we are seeking.

1.3 What is automatic control?

Control theory was developed to support the emergent activity of automatic control. It is therefore a useful motivation to turn our attention to automatic control. Historically, the discipline of automatic control was concerned with the replacement of the human worker of Figure 1.7 by the automatic controller of Figure 1.8.

Although automatic control is nowadays a complex discipline, no longer primarily concerned with the replacement of human operators, it is a useful starting point to consider what sort of skills are necessary to move from an existing, manually controlled situation to a new automatically controlled situation, as in Figure 1.8.

(1) A central idea of control theory is the control loop. All control loops have the same basic form, regardless of the particular application area. Thus, control theory uses an application-independent notation to convert all control problems into the same standard problem. We can consider that control theory concentrates on studying the universal situations that underlie all applications of quantitative control. In broadest form, a control loop appears as in Figure 1.9. The decisions

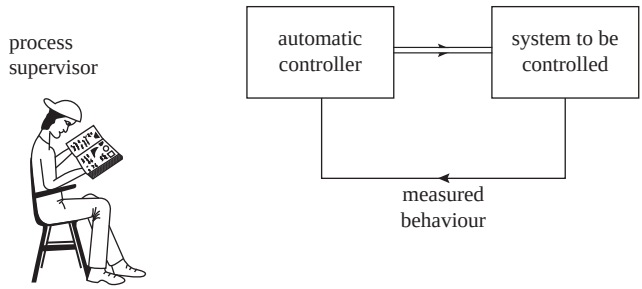


Figure 1.8 The process of Figure 1.7 now under automatic control

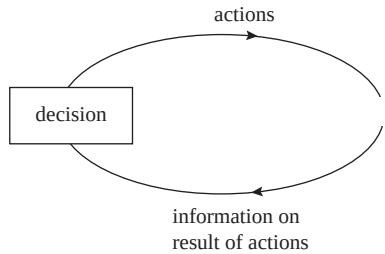


Figure 1.9 A control loop in its broadest form

govern actions that are taken. The effect of the actions is reported back by the information channel. Further decisions are taken and the loop operates continuously as described. A control loop provides an extraordinarily powerful means of control but, at the same time, the existence of the loop always brings the possibility of the potentially very destructive phenomenon of instability.

(2) All control loops are error driven, where error is defined as the difference between the behaviour that is desired and the behaviour that is measured.

(3) An important performance measure for a control system relates to rate of error reduction. Often, performance is quoted in terms of the highest frequency that the control system can follow, when required to do so.

(4) All control loops tend to become unstable as higher and higher performance is sought. A good understanding of the topic of stability is central to understanding control theory.

1.4 Some examples of control systems

Four control systems are illustrated in Figure 1.10. All can be seen to have the form of Figure 1.11. A user, uninterested in the mechanics of all this, will see the simpler view of Figure 1.12. We refer to this single block (that has the control loop hidden inside) as the control system. The following further points are important:

(5) Control system performance can only be meaningfully specified in relation to the (total) control system of Figure 1.12.

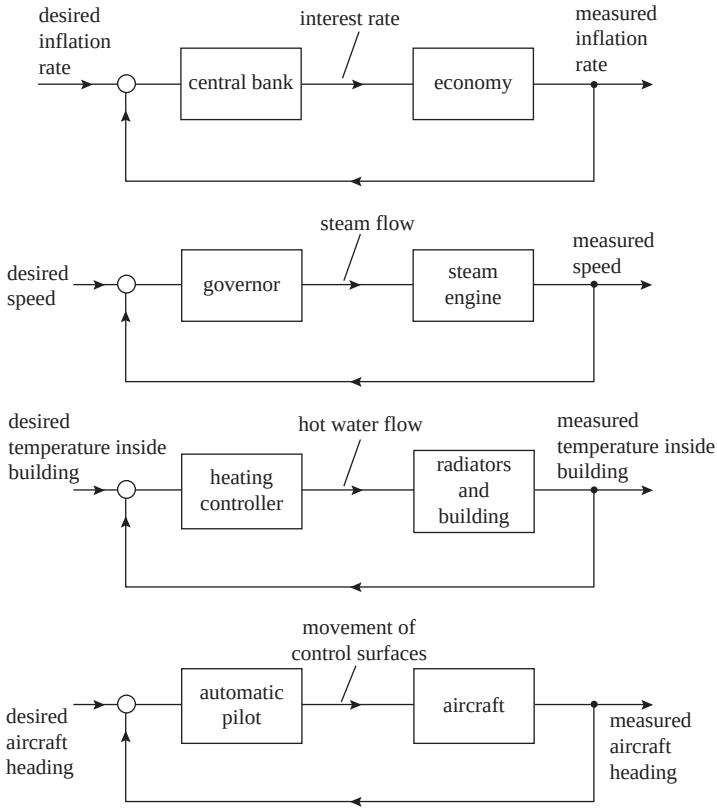


Figure 1.10 Some examples of particular control applications

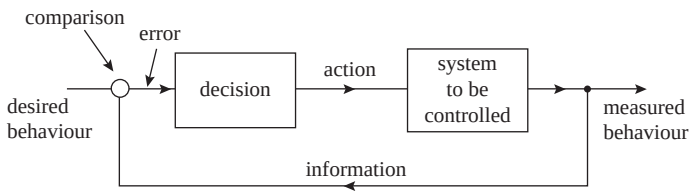


Figure 1.11 The general form of all the control systems in Figure 1.10

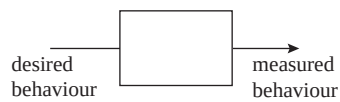


Figure 1.12 A user's view of the control system of Figure 1.11

(6) The control system designer almost always has to incorporate into the control loop an element whose intrinsic behaviour is largely outside his own influence. (For instance, the control systems designer may have little influence on the design of a building although later this person will be called upon to design temperature control systems for it.)

(7) To quite a large extent, the controller must neutralise adverse characteristics in the process, compensating for non-ideal process configurations and for short- and long-term perturbations and variabilities.

(8) For (7) to be possible, the process characteristics must be known to some degree of accuracy and be reasonably constant.

(9) Ideally (see (6)) the control system designer will ensure that the process has the best possible inherent behaviour, even with no control. The control design cycle therefore roughly includes the following steps:
   (a) Decide on a necessary performance specification.
   (b) Quantify the performance of any system-to-be-controlled element that is to be included in the control loop.
   (c) Design, by one or other control design techniques, a controller so that the control system meets the specification of (a).
   (d) Construct, commission and test the control system.

In the next chapter, we take these ideas further.

Chapter 2

Control design ideas: a non-mathematical treatment

2.1 Initial discussion

In the previous chapter we saw that prerequisites for control design were broadly as follows: a defined objective, a set of available actions and a model that could be interrogated to establish which of the available actions would best move the system towards meeting the objective. Now we add more structure to the concepts to put forward a possible design methodology (Figure 2.1). In this methodology, central use is made of a system model. This model is assumed to be able to rapidly calculate the expected behaviour of the system when subjected to any particular action.

Questions that immediately arise are as follows:

- In practice can a realistic model be produced? If so how?
- By what mechanism can two sorts of behaviour be compared?

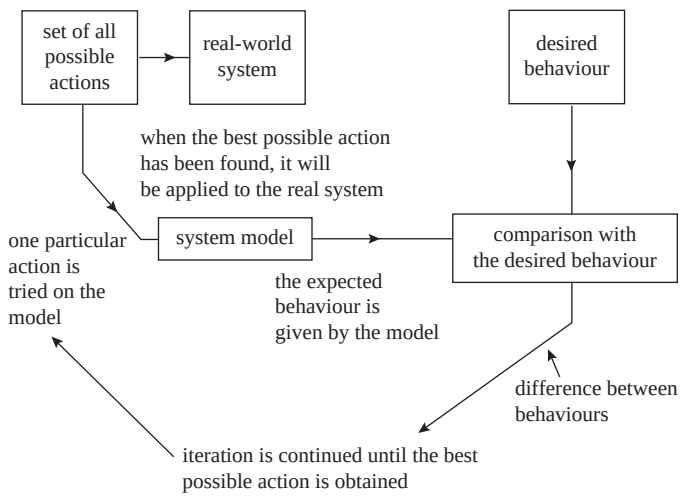


Figure 2.1 A possible methodology for control system design

- Can the difference between desired behaviour and expected behaviour be meaningfully used to help the iteration towards the best possible choice of action?
- How fast would the iterative procedure, involving the model, have to operate in order for the real-world system to be realistically controlled?

We answer none of these questions directly, preferring to state that Figure 2.1 remains largely symbolic. Meanwhile we ask a further question.

2.2 Question: Can the best possible control actions be synthesised by some mechanism?

If the system model and the desired behaviour are accurately defined, should it not be possible, in one pass, to synthesise the necessary actions shown in Figure 2.1 without interactive searching? This question is illustrated graphically in Figure 2.2.

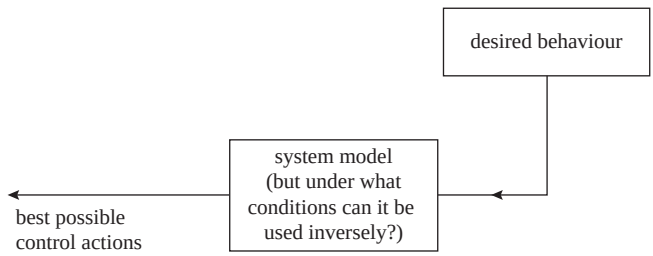


Figure 2.2 The idea of using a system model inversely to synthesise actions

2.3 Requirements for an automatic control system

If it is possible to synthesise the best possible actions continuously by some sort of algorithm, then we have arrived at automatic control.

In the best known and simplest form of automatic control, the desired behaviour is specified as a requirement that the measured system response (say y) should continuously and closely track a required system response (say v) that is input by the system user (Figure 2.3).

Of course, v may be constant or even always set equal to zero. In such cases, an automatic control system has the task of keeping a measured value of y always equal to the specified constant value of v, despite the presence of disturbing influences. These general requirements of an automatic control system are shown in Figure 2.4. Moving more towards the realisation of a practical system, Figure 2.5 results.

It is clear that the success of the scheme presented in Figure 2.5 depends on the disturbances w being measurable and on the existence of an accurate quantitative understanding of the system to be controlled, for otherwise the 'generator of control actions' cannot be accurately constructed. (Notice that no use is made of any measurement of the response.)

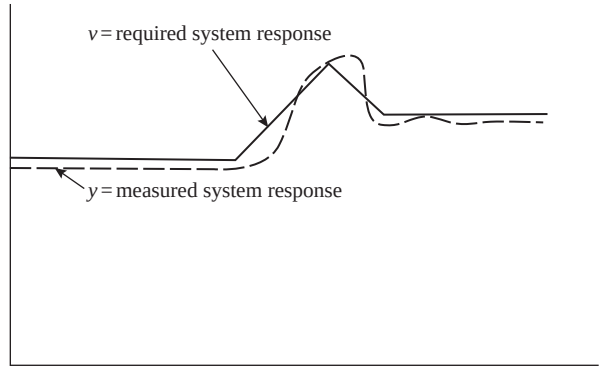


Figure 2.3 An automatic control system may be required to force the measured response y *to track a user-specified desired response as closely as possible*

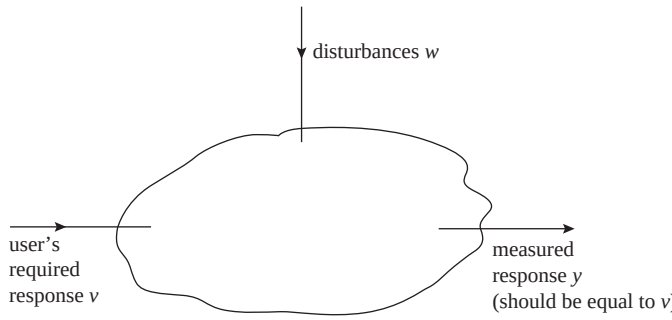


Figure 2.4 Requirements for an automatic control system

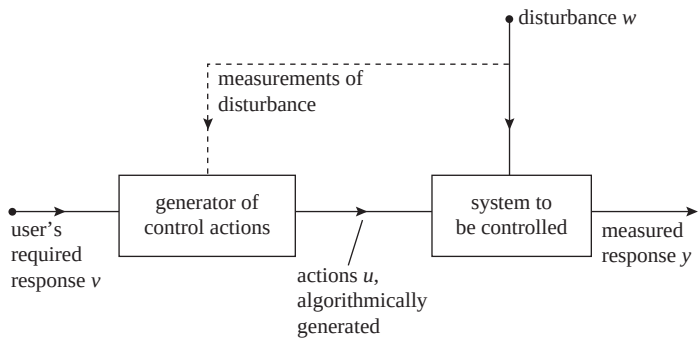


Figure 2.5 Realisation of an automatic control system

2.4 Automatic feedback control

Automatic feedback control overcomes both the above problems (possible unmeasurability of disturbances, difficulty of obtaining a sufficiently accurate model) by being error driven as shown in Figure 2.6.

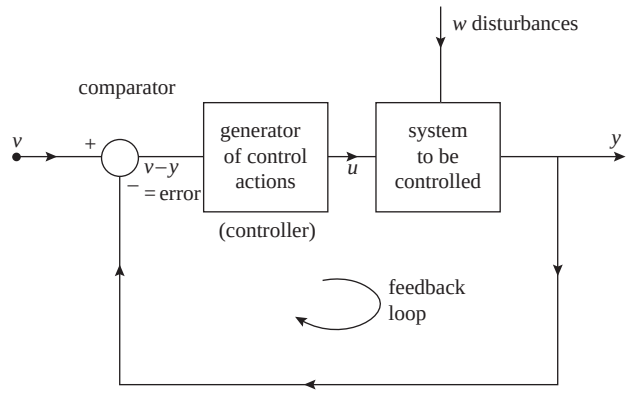


Figure 2.6 An 'error-driven system': the feedback loop

2.5 Diagrams illustrating and amplifying some of the concepts described so far and showing relationships to a software engineering context

(1) Control theory is interested in systems behaviour and deals with generalised situations called systems. A system is a set of elements interconnected by information links and existing within a system boundary outside which is the system environment. Figure 2.7 illustrates some of the rationale.
(2) A broad task is to go from a statement of 'desired behaviour' to the synthesis of a system exhibiting that desired behaviour (Figure 2.8).
(3) In more specific terms, control theory is first concerned with systems understanding, second with influencing systems behaviour and third with designing systems to exhibit particular behaviours (Figure 2.9).

Almost every important application of control theory is closely embedded within a complex software engineering context. Without attempting to go into details, the following concept diagrams illustrate some of the interactions between control design approaches and the software context:

(4) Once systems behaviour is considered, the following questions arise: What types of behaviour do we have in mind? How can behaviour be quantified? What factors limit performance? (Figure 2.10).
(5) Elaborating on the points in Figure 2.10, we turn to points of methodology. How can we find out what type of system is really required? How can we turn this knowledge into a specification and then into a design? What tools are available to assist us? Figure 2.11 illustrates these points.

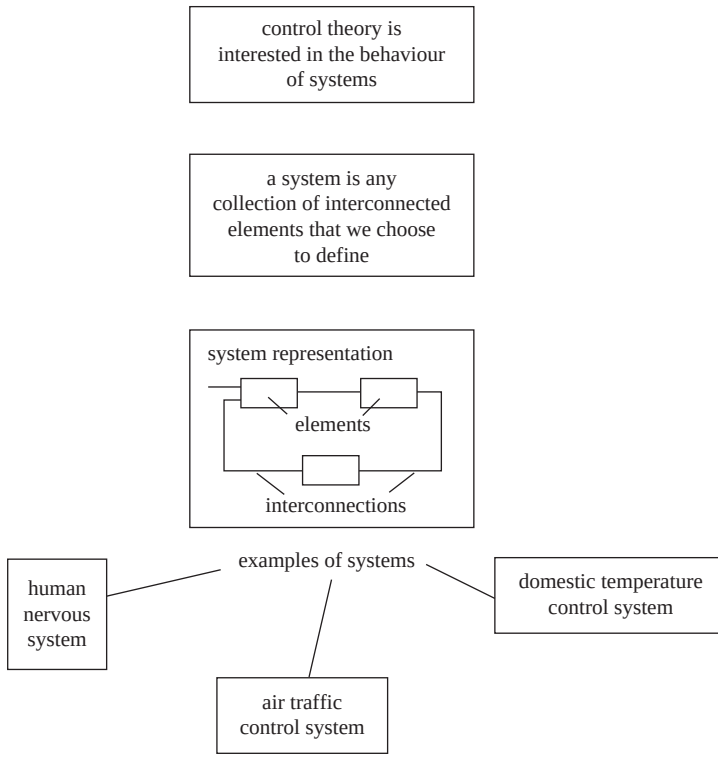


Figure 2.7 Some basic control ideas

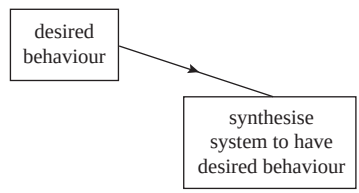


Figure 2.8 The broad task of control design

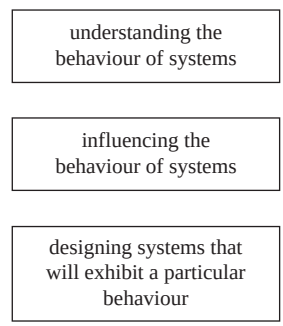


Figure 2.9 The sequence of objectives involved in a typical control project

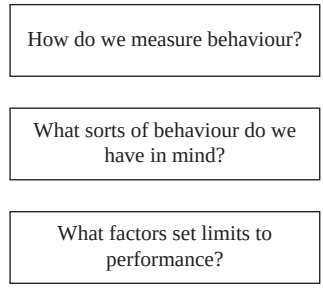


Figure 2.10 Fundamental questions related to system behaviour and system performance

(6) Elaboration of the points in Figure 2.11 produces Figure 2.12. Here we see a stage called 'requirement capture', dedicated to establishing what the eventual user needs. Further stages of systems specification, system design, knowledge elicitation (aimed at feeding in particular expert knowledge) and data base design precede the writing of code (i.e. programming) and the proving, commissioning and maintenance that are essential parts of all real applications.

(7) Figure 2.13 is a re-run of Figure 2.12 with a few enhancements. This figure illustrates how a user's conception of the ideal system is modified by additional enhancements as well as by restrictions suggested by a systems designer's expertise. The role of CASE (computer-aided software engineering) tools can be seen in the diagram. These tools allow systematic top–down design, partitioning of work tasks into manageable parcels, continuous checks on consistency and a graphical overview of the whole design project. The figure also illustrates how so-called reverse engineering is used to check that the final codes are in complete and consistent agreement with the initial system specification.

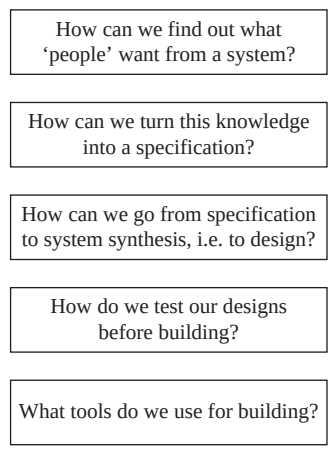


Figure 2.11 The beginnings of a methodology for system design

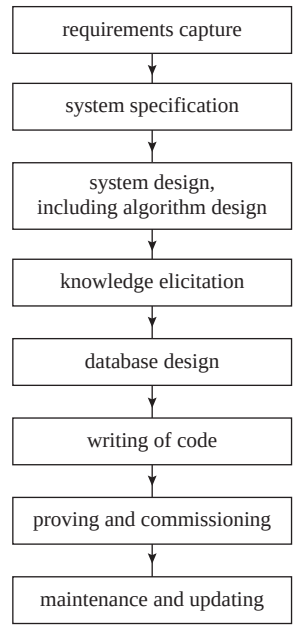


Figure 2.12 System design from requirements capture to commissioning and maintenance

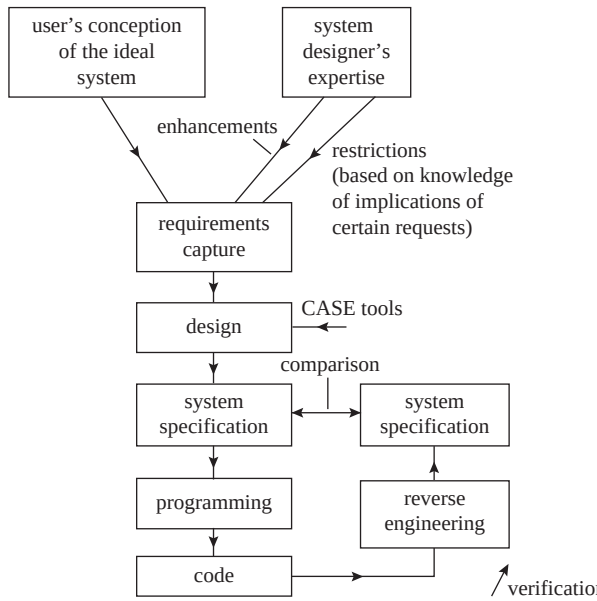


Figure 2.13 A more detailed view of system design showing the role of CASE tools and the place of verification using reverse engineering

Chapter 3

Synthesis of automatic feedback control loops: a more quantitative view

3.1 Feedback loops: further discussion

In automatic control, a device called a controller issues commands that are physically connected to a process with the intention to influence the behaviour of the process in a particular way. The commands that will be issued by the controller in a particular set of circumstances are completely determined by the designer of the controller. Thus, automatic control can be seen to be completely pre-determined at the design stage.

The controller may be driven by time alone or it may be driven in a more complex way by a combination of signals. In feedback control, the controller is error driven. That is, the controller receives a continuous measurement of the difference between required behaviour and actual behaviour and its output is some function of this error (Figure 3.1).

In this type of system, excellent results can be obtained in practice with very simple controllers indeed, even when operating under conditions where the system to be controlled is not well understood. Roughly speaking, we can imagine that the controller will keep on taking corrective action until the error is reduced to zero.

An alternative view of the arrangement of Figure 3.1 is that users see an artificially enhanced system that has been synthesised to meet their wishes. If we represent the controller by an operator D and the system to be controlled by an operator G, we obtain:

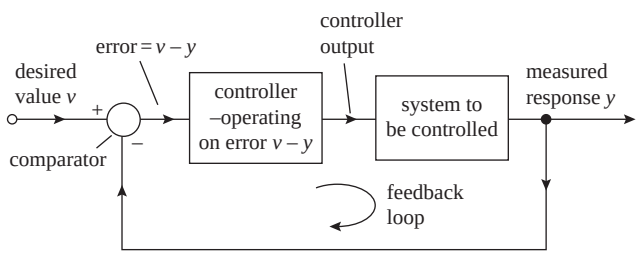


Figure 3.1 A feedback control loop. Notice that the output of the controller is a function of error v – y

$$\left.\begin{array}{r} \text{System output} = Gu \\ \text{Controller output} = De \\ \text{Controller input} = e = v - y \end{array}\right\} \tag{3.1}$$

The design task is to specify the controller D, connected to a given process G as in figure 3.1, such that a satisfactory overall performance is obtained. We can imagine that the controller modifies the process characteristics in ways chosen by the designer.

We next assume that there exists a desired hypothetical process H. By suitable connection of a controller D to the actual process, are we able to produce a configuration that behaves the same as H?

If we interconnect G and D as shown in Figure 3.2 and assume some benevolent mathematics that allows us to manipulate the symbols, then from the figure,

$$y = GD(v - y)$$

$$\frac{y}{v} = \frac{GD}{1 + GD} \tag{3.2}$$

and setting

$$D = \frac{H}{G(1 - h)} \tag{3.3}$$

will be found to accomplish the objective of making y/v equal to H. In other words, this choice of D does indeed make the synthesised configuration behave like the chosen hypothetical process H.

Here we assume that well-behaved operators can be found to operate on the sort of functions that exist in the control loop and possess those other properties of associativity and invertibility that are needed to make manipulation valid (i.e. we assume that the operators G, D, H are elements in a group).

Laplace transforms or other techniques can produce these operators for specific examples, but for the moment, it is sufficient to know that such operators exist.

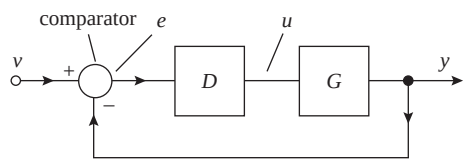


Figure 3.2 A feedback loop with the system to be controlled denoted G and the controller denoted D

Then, from the set of equations above, it is clear that

$$y = [(1 + GD)^{-1} GD]v \tag{3.4}$$

and the system represented by the operators in the square brackets can be synthesised by choice of D to behave as the user requires.

We note and ask:
D contains G^{-1}, the inverse of the plant:

- This may be of high order.
- Is it (G) known?
- Does it (G) stay constant?
- If G changes by (say) 10%, will control become very poor?

(i) Can our requirements be adequately represented by an operator H?
(ii) How is H chosen?
(iii) Is it not disturbing that H is not in any way dependent on G? For instance, can we turn a low-performance aircraft (G) into a high-performance aircraft (H) simply with the aid of a clever algorithm? A simple question, but it leads to a valuable conclusion: high performance can rarely be obtained algorithmically but almost always requires expensive equipment such as big engines and high power-to-weight ratios.
(iv) Does D turn out to be a possible, buildable, robust, practical controller?

Comment
Limits on attainable performance are set by the constraints in the process. These constraints are not at all modelled by the (linear) operator G, nor are they otherwise easily fed into the design procedure.

A key point is that if H is chosen too ambitiously, then D will simply drive the process G into saturation.

In practice, a particular process G can nearly always be marginally improved to (say) a faster responding H, whereas it will rarely be able to be improved by several orders of magnitude. *The chief difficulty therefore lies in specifying H – how ambitious can we be?*

3.2 What sorts of control laws are there?

It would appear reasonable that an infinite variety of control laws might be possible including some highly exotic versions that would need considerable computer power for their implementation. However, we shall show that if the control law is restricted to be linear, then the range of possible control laws is very restricted indeed.

Without much loss of generality, we may assume that the control law is to be implemented by an idealised computer that occupies the ‘controller’ position in Figure 3.1.

The output of the controller at any instant of time can be any function of the current and/or previous error signal that is read into the controller. (Recall that the

system is operating in real time and that, therefore, future values of error cannot, by definition, be available to the controller.)

If linearity is now insisted on in the controller, then the possible control laws are severely restricted to be of the form:

Present output of the controller = some multiple of the present input
+ multiples of the previous inputs
+ multiples of the previous outputs

In other words, the present output of the controller is constrained to be just a weighted sum of present and past values of the input to and the output from the controller.

Corollaries

(i) More 'intelligent' control laws may contain models of the process to be controlled, and using these models, for instance, in rapid iterative simulation mode, they may calculate and produce a control output. Such control laws are not linear and theoretical calculation of their expected performance is therefore a difficult task.

(ii) The restricted class of control laws that can be implemented linearly excludes many optimal control strategies. This is why so often optimal control solutions appear as pre-specified (open loop) functions of time that cannot be converted into automatic feedback controllers except in a minority of cases.

(iii) Specifically, non-linear controllers have found very little application. This is surprising since most processes that have to be controlled are fairly non-linear and it would seem that non-linearity in the process could surely be cancelled by 'opposing' non-linearities in the controller to give overall good control. Also, nature is a well-known user of non-linear devices in most of its control applications, for instance, in the human body, and we might reasonably expect control design to follow in this direction.

3.3 How feedback control works: a practical view

The illustrations use temperature control and foreign currency exchange control but the results are valid for any feedback loop.

Block G (Figure 3.3) is a heating process. It receives an input of 'fuel flow' and produces an output 'temperature'.

Block D (Figure 3.4) is a motorised fuel valve. When the control signal is zero, the valve produces a fuel flow u_0. When the control signal is positive, the fuel flow is increased as shown in Figure 3.5. The larger the control signal, the steeper the rate of increase (Figure 3.6). Conversely, negative control signals produce decreasing fuel flows.

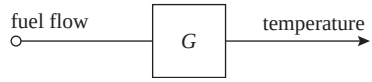


Figure 3.3 A heating process viewed as an input–output device

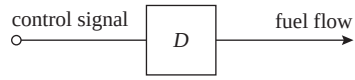


Figure 3.4 A controller for connection to the heating process

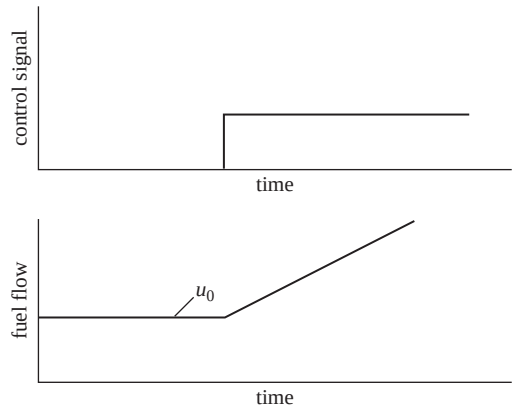


Figure 3.5 The characteristic of the controller D: how the control signal causes changes in fuel flow

If now the feedback loop in Figure 3.7 is formed, the input to the motorised valve *D* is the difference between the temperature that is desired and the actual (measured) temperature.

Assume that the measured temperature is 80° C and the desired temperature is 100° C. Then the input received by the valve *D* will be $100 - 80 = 20$. This is a positive signal and valve *D* will respond by increasing the fuel flow. Heating process *G*, on receiving an increased fuel flow, will respond by increasing its temperature so that it will climb above 80° C. The error will decrease and the fuel flow will settle eventually at the value that brings the measured and desired temperatures to be equal, i.e. to a zero error condition. The operation just described is illustrated in Figure 3.8.

Notice carefully that the temperature will arrive exactly at the desired value regardless of the particular characteristics of heating process and valve. For instance, even should the heating process suddenly and unexpectedly fall in efficiency (thereby requiring more fuel to achieve the same temperature), the feedback loop will compensate perfectly for this change since the fuel flow will be increased automatically

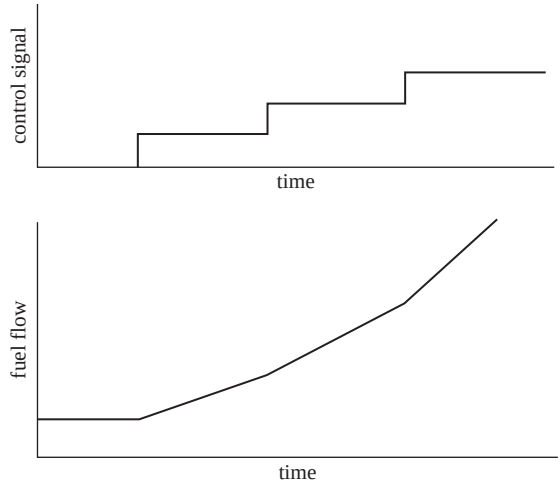


Figure 3.6 Further illustration of the characteristics of the controller D. How stepwise increases in control signal are translated into increasing rates of fuel flow

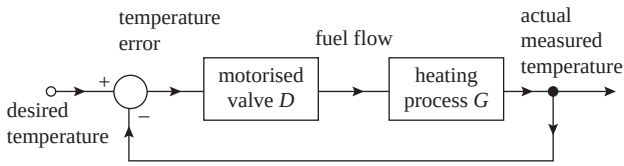


Figure 3.7 A feedback loop in which the motorised valve is connected to the heating process

to whatever level is required to give exactly the desired temperature. Here we see the great attraction of feedback control – an imperfectly understood process, even one subject to large unpredictable changes of basic characteristics, can be satisfactorily controlled using a control law that is specified in the vaguest of terms.

Before we move on to consider the implications, let us illustrate the feedback control principle at work in a different, much wider context (Figure 3.9). Here let the element *G* be an economic element whose input is UK bank rate (%) and whose output is the exchange rate, number of US dollars per pound sterling.

Assume that the Bank of England has in mind a desired exchange rate, say $1.6 against the pound. It is 'generally accepted' that increasing the UK interest rate will increase the exchange rate. The Bank, *D*, in the feedback control loop, therefore manipulates the interest rate to whatever level is necessary to achieve the desired exchange rate (Figure 3.10).

Of course, the Bank does not ramp the exchange rate (as in the earlier fuel rate example) – but rather moves it in a succession of steps to form a staircase function that is all too familiar (Figure 3.11). Notice again that (fortunately) the Bank does

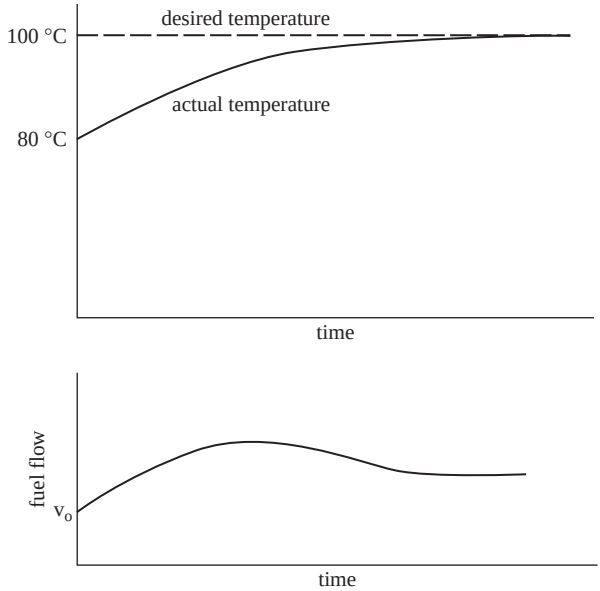


Figure 3.8 Expected behaviour of the heating process when under closed loop control

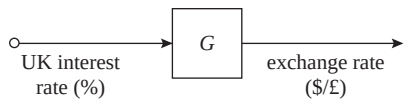


Figure 3.9 The economic element that relates exchange rate to UK interest rate

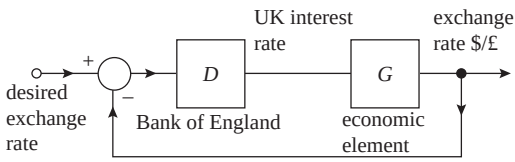


Figure 3.10 The economic element under a loose form of closed loop control by the Bank of England

not need to understand how the economy works to attain the required exchange rate, using the principle of feedback.

The feedback principle works extremely well provided that the available actions do not encounter constraints that limit their magnitudes. In the case of temperature control, there will always be some limit on fuel flow rate. In the case of exchange rate control, there will always be restraints, often of a political nature, on the magnitude of the interest rate that can be used. Linear systems have no such

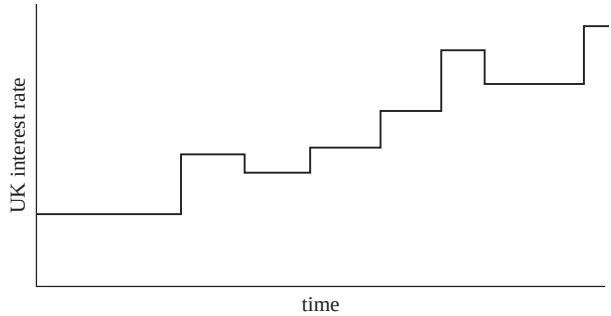


Figure 3.11 A typical interest rate profile resulting from the Bank of England's actions

constraints and hence linear control theory can never deal satisfactorily with the inevitable boundedness of all real control actions.

We now return to the main theme of practical feedback control. We recall that the approach has the considerable merit that it offers exact control of vaguely specified and possibly changing mechanisms, using quite loosely specified control actions. The underlying rough idea is that the action in the control loop keeps on increasing/decreasing to whatever level is needed to make the error zero. So long as the error is non-zero, further action is taken in the direction that will reduce the error. When the error reaches zero, the value of the controlled variable is, by definition, equal to the specified desired value.

We have seen that an acceptable level of control can be obtained for imperfectly understood processes using vaguely specified actions. However, it is now time to ask:

(i) How long does it take for control to be achieved and what is the nature of the response curve?
(ii) Can a 'best possible response' be defined and, if so, how can it be achieved?
(iii) In a particular case, what sets the limit on performance?
(iv) What if the desired target is not constant (a moving target) or there are external influences outside our control?

(i) Responses may range across the type of behaviour shown in Figure 3.12. It is clear that, for many applications, the nature of the response and the time taken to achieve control will be critical, yet these aspects cannot be predicted in the absence of quantitative data.
(ii) A 'best possible response' is only meaningful in general for problems where constraints are present. By definition, these problems do not belong to linear control theory.

Linear control systems can, by definition, use signals of any magnitude to produce responses that, in the limit, are instantaneous – such responses are clearly unattainable in practice and will be prevented by (unmodelled) constraints or (unmodelled) noise entering the argument. That is to say, either noise or non-linearity will eventually set

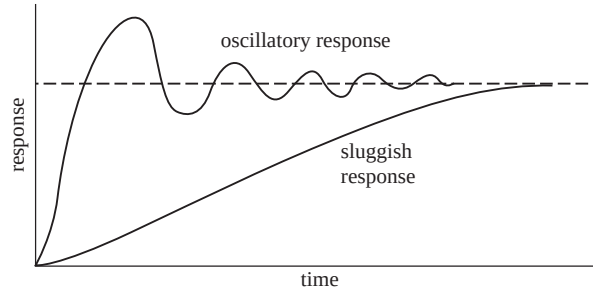


Figure 3.12 Typical transient responses ranging from highly oscillatory to sluggish

limits to attainable performance. The difficulty is overcome in practice as follows. A required response that is realistic for the application but that is not expected to violate constraints is aimed for. If this rather empirical approach shows that constraints would be violated, the problem has to be altered. In an engineering application, more powerful motors, stronger practical components or additional amplifier stages may be needed.

The valuable point emerges: The limits of control performance are the constraints within the system; and these are not at all represented in linear control theory.

We have now reached the stage where 'imported detail' begins to crowd in on us, attempting to force us away from principles into a discussion of technique. At this point we are content to say that, even under conditions of moving targets, external influences and other factors yet to be discussed, viable feedback control systems can usually be designed and implemented.

3.4 General conditions for the success of feedback control strategies

By the nature of feedback control, corrective action can only begin once an error has been detected. Therefore, close control will only be possible in those cases where the rate of corrective action can at least match the rate of disturbance generation. This idea, of course, soon leads to requests for high bandwidth of control loops to allow, in one way of looking at it, the control loop to successfully synthesise a signal equal and opposite to the disturbance signal (Figure 3.13).

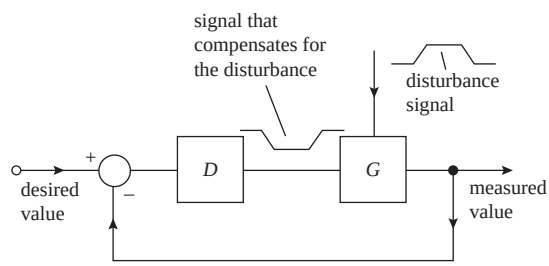


Figure 3.13 An (ideal) feedback controller will synthesise an equal and opposite signal to neutralise the effect of an incoming disturbance

In many cases, it is not possible to design a closed loop with a high enough bandwidth, and then feedback control has to be abandoned or relegated to a secondary role.

3.5 Alternatives to feedback control

Following are alternatives to feedback control:

(i) *Pre-programmed control*: Here a standard strategy, recipe or sequence of controls is calculated in advance and is implemented without regard to any signals that come from the system during the period of control.

(ii) *Feed-forward control*: Here the disturbing signals are measured and necessary corrective actions are calculated and implemented with the idea of eliminating error before it can occur. This approach requires that the disturbances are measurable independently (as opposed to the feedback approach that allows the error to be a measure of received disturbances) and that the necessary control actions are accurately calculable.

(iii) *Prediction followed by control*: Here prediction of future conditions, based either on extrapolation algorithms or on stored historical records, is used to allow the best possible positioning of a low-bandwidth control system. A classical case is in electricity generation where rapidly changing consumer demand follows a reasonably predictable daily and seasonal pattern, thereby allowing the cumbersome process (time constant of several minutes) of bringing new generators onto the grid to be scheduled to match load predictions rather than attempting an unsuccessful feedback control in which the slow process of bringing new generators on-stream attempts to match the very much faster rate of change of consumer electricity demand.

Note: Source material and suggestions for further reading to support the topics of this chapter will be found in Chapter 18.

Chapter 4

How the Laplace transform greatly simplifies system representation and manipulation

4.1 Laplace transform techniques

Many useful techniques depend on the Laplace transform. The Laplace transform of a function $f(t)$ is denoted sometimes by $\mathcal{L}\{f(t)\}$ and sometimes by $F(s)$. The inverse Laplace transform of $F(s)$ is denoted sometimes by $\mathcal{L}^{-1}\{F(s)\}$ and sometimes by $f(t)$. Figure 4.1 makes the relation clear; s is a complex variable whose role is defined by (4.1).

4.2 Definition of the Laplace transform

By definition

$$\mathcal{L}\{f(t)\} = \int_0^\infty \exp(-st) f(t)\, dt \tag{4.1}$$

Examples

(1) Let $f(t) =$ a constant k, and let R(s) denote the real part of the complex number s

$$\mathcal{L}(k) = \int_0^\infty \exp(-st) k\, dt = -\frac{1}{s}\exp(-st)\Big|_0^\infty = 0 - \left(-\frac{k}{s}\right) = \frac{k}{s} \tag{4.2}$$

provided that R(s) is positive (for otherwise the integral does not exist).

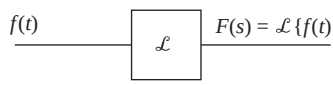


Figure 4.1 The Laplace transform operation

Let $f(t) = \exp(at)$

$$\begin{aligned}\mathcal{L}\{\exp(at)\} &= \int_0^\infty \exp(-st)\exp(at)\, dt \\ &= \frac{1}{(a-s)}\exp(a-s)t\Big|_0^\infty \\ &= \frac{1}{s+a}\end{aligned}$$

This will be true provided that $\mathrm{R}(s) > a$.

The chore of calculating Laplace transforms of particular time functions and the converse problem – calculating the time function, by inverse Laplace transformation, corresponding with a particular Laplace transform – can be avoided by the use of readily available software packages or tables of transform pairs such as Mobile Reference (Kindle Edition, 2009). Small tables are to be found as appendices in many introductory control textbooks. A very satisfying slim volume of 520 transform pairs for the desktop is McCollum and Brown (1965) and a very comprehensive set in Prudnikov *et al.* (1992).

4A Convergence of the integral that defines the Laplace transform

It is quite typical, as in the last example, for the integral that defines the Laplace transform to be finite (and hence defined), only for restricted values of s. However, there seems to be a tacit agreement in the teaching of control theory to avoid any discussion of the distracting question: What is the significance of the region of convergence of the integral that defines the Laplace transform?

For example, let $a = 2$ in the transform $1/(s + a)$ that we have just derived. Then it is clear that the transform is only defined and valid in the shaded region in Figure 4.2 where the real part of s is strictly greater than 2.

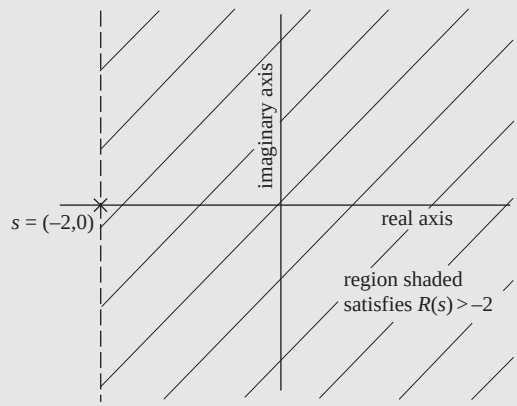


Figure 4.2 *The transform $1/(s + 2)$ is only defined in the shaded region, yet the point $s = (-2, 0)$ is the one of interest and the transform is universally used at that point without further question*

However, later in this chapter, we shall see that, for this transform, the value of s for which $s + a = 0$ is highly significant (i.e. the point $s = (-2, 0)$). We, in common with the whole control fraternity, blithely use the transform at the point $s = (-2, 0)$ where it is undefined.

Notice also that the region in which the integral converges may be empty. For example, the function $\exp(t^2)$ has no Laplace transform for this reason.

4B Problems with 0^- and 0^+

(i) Anyone who has used Laplace transforms to solve differential equations will be used to obtaining solutions such as

$$y(t) = y(0)\exp(-t)$$

where by $y(0)$ is meant $y(0^+)$, which has to be calculated independently. One is expected to know $y(0^+)$, but $y(0^+)$ is really part of the solution that is to be determined. Clearly $y(0^+)$ will be different from $y(0^-)$ only when there is a discontinuity at the origin. Such a situation occurs for instance in calculating the step response of a system containing a differentiator. The difficulty can sometimes, but not always, be overcome by exercising common sense.

(ii) A rigorous examination of the Laplace mechanism applied to a delta function unearths problems again due to the 0^-, 0^+ phenomenon. Taking the phenomenon rigorously into account shows that $\mathcal{L}(\delta(t)) = 0$, rather inconveniently, compared with $\mathcal{L}(\delta(t)) = 1$, that we universally use. The indispensable reference Zadeh and Desoer (1963) discusses the Laplace transform rigorously.

4.3 Use of the Laplace transform in control theory

Consider a system (Figure 4.3) that receives an input $u(t)$ and in response produces an output $y(t)$. The response $y(t)$ is determined by the nature of the input signal $u(t)$ and by the nature of the system.

Suppose that $g(t)$ is the response of the system to a unit impulse applied at time $t = 0$. Then the response to any other input u is given by the convolution integral (see interlude 4C for further insight).

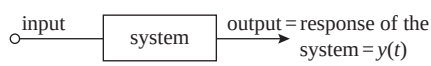


Figure 4.3 A simple input/output system

However, life is much simpler if we use the Laplace transforms of $u(t)$ and $g(t)$ to yield $u(s)$, $G(s)$, respectively, for then, equivalent to (4.2), we have

$$y(s) = G(s)u(s) \tag{4.3}$$

i.e. transform-domain multiplication is equivalent to time-domain convolution.

There is an additional advantage in that inverse transformation from $y(s)$ back to $y(t)$ is often not required – many interesting and significant questions can be answered most efficiently by reference directly to $y(s)$. The equivalence between (4.2) and (4.3) is very significant. Refer to Section 4.4 for an alternative viewpoint. Refer to Dorf (2011) for a more detailed derivation.

4.4 The concept of transfer function

The transfer function of a dynamic system with input $u(t)$ and output $y(t)$ is defined to be the Laplace transform of $y(t)$ under the condition that $u(t)$ is a unit impulse applied at time $t = 0$; or, more generally applicable in practice:

$G(s) = y(s)/u(s)$, valid for any u, y pair whose transforms exist.

Consider next the interconnected systems shown in Figure 4.4. Let the two systems have impulse responses $g_1(t)$, $g_2(t)$, respectively.

$$\begin{aligned} \text{Then } y(t) &= \int_0^t g_2(t-\tau)u(\tau)\ d\tau \\ &= \int_0^t g_2(t-\tau) \int_0^\tau g_1(t-p)v(p)\ dp\ d\tau \end{aligned} \tag{4.4}$$

However, using Laplace transformed signals and transfer functions (i.e. Laplace transformed impulse responses), we obtain, instead of (4.4),

$$y(s) = G_2(s)G_1(s)v(s) \tag{4.5}$$

4.5 System simplification through block manipulation

Block diagrams of any size and complexity can always be reduced to a single block by successive application of three rules that are summarised in Figure 4.5. The rules are easily derived as follows (rule 3 of Figure 4.5):

$$e(s) = v(s) - y(s), y(s) = G(s)e(s)$$

$$y(s) = G(s)v(s) - G(s)y(s), y(s)(1 + G(s)) = G(s)v(s)$$

$$y(s) = \frac{G(s)v(s)}{1 + G(s)}$$

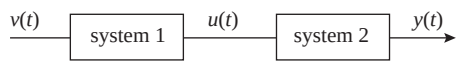


Figure 4.4 Two systems connected in series

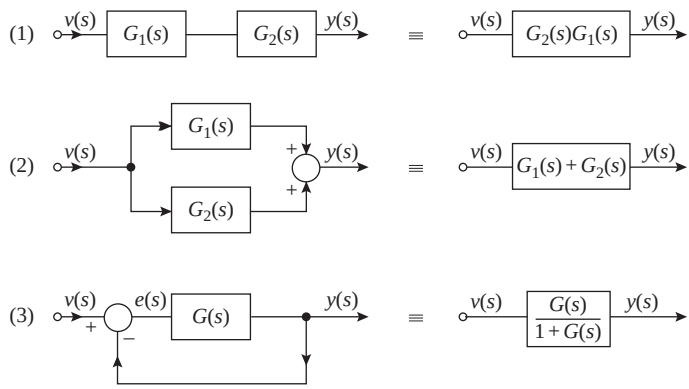


Figure 4.5 Three basic configurations and their equivalent single-block representations

Complicated block diagrams can with advantage be reduced with the aid of Mason's rules (see Dorf, 2011).

4.6 How a transfer function can be obtained from a differential equation

If a differential equation

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots = b_r \frac{d^r u}{dt^r} + \cdots$$

is Laplace transformed, we obtain

$$(s^n + a_{n-1}s^{n-1} + \cdots)y(s) + \text{terms depending on initial conditions} = (b_r s^r + \cdots)u(s) + \text{terms depending on initial conditions}$$

Transfer function analysis, but note not differential equation solution by Laplace transforms, assumes that initial condition effects have died away and that the output is a function of the input only. In that case, the transfer function corresponding to the differential equation is

$$\frac{y(s)}{u(s)} = \frac{b_r s^r + \cdots}{s^n + a_{n-1}s^{n-1}}$$

4.7 Poles and zeros of a transfer function

Any value of the complex variable s for which $G(s) = 0$ is called a zero of $G(s)$. Any value p of the complex variable s that satisfies $s \to 0 \Rightarrow G(s) \to \infty$ is called a pole of $G(s)$.

If $G(s)$ can be expressed as $G(s) = P(s)/Q(s)$, then the zeros are the roots of the equation $P(s) = 0$ while the poles are the roots of the equation $G(s) = 0$. In a pole–zero diagram, zeros are denoted by the symbol 0 and poles by the symbol $\times$ in the complex plane.

The mathematical underpinning of the theory of transfer functions is provided by complex variable theory. Particularly relevant aspects of complex variable theory are Cauchy's integral theorem and Cauchy's integral formula, Laurent series and the associated concept of residues. [These aspects can be pursued in Aggarwal *et al.* (2011).]

4.8 Understanding system behaviour from a knowledge of pole and zero locations in the complex plane

The system to be investigated (Figure 4.6) has a single input u and a single output y. Suppose the transfer function of the system is $G(s) = P(s)/Q(s)$, where P, Q are polynomials with real coefficients in s. Since

$$y(s) = G(s)u(s) = \frac{P(s)}{Q(s)}u(s)$$

we can write

$$Q(s)y(s) = P(s)u(s)$$

Evidently $Q(s)$ governs the nature of the system's response to initial conditions and hence also its stability (since a response to initial conditions that dies away to zero belongs to a stable system and a response to initial conditions that grows with time belongs to an unstable system).

Conversely, $P(s)$ affects the manner in which the system responds to external inputs.

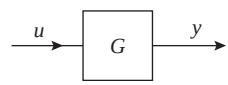


Figure 4.6 A simple input/output system

4.8.1 Meaning of pole locations

Figure 4.7 summarises some of the most important points related to the following question: What is the relation between the transfer function pole locations in the complex plane and the time-domain behaviour of the system?

Figure 4.7(a) shows how the rate of change of transient solution increases as the pole to origin distance increases; Figure 4.7(b) shows how any pole in the right half plane indicates instability; Figure 4.7(c) shows the split of the complex plane into the real line (poles on the real line indicate exponential responses) and the

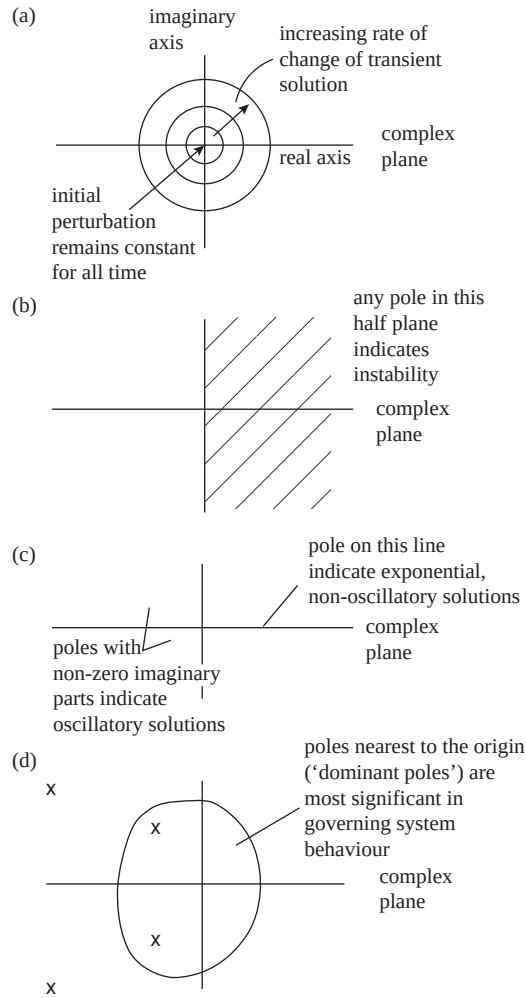


Figure 4.7 The meaning of pole locations

remainder (when poles indicate oscillatory responses) and Figure 4.7(d) shows how poles nearest to the origin 'dominate' the response.

Zeros also have an effect on system response. Figure 4.8 gives examples of pole–zero diagrams and their associated system step responses.

4.9 Pole placement: synthesis of a controller to place the closed loop poles in desirable positions

Suppose a given system G has poles as shown in Figure 4.9, but it is required that the poles are actually at the positions shown in Figure 4.10. Then, preceding the given

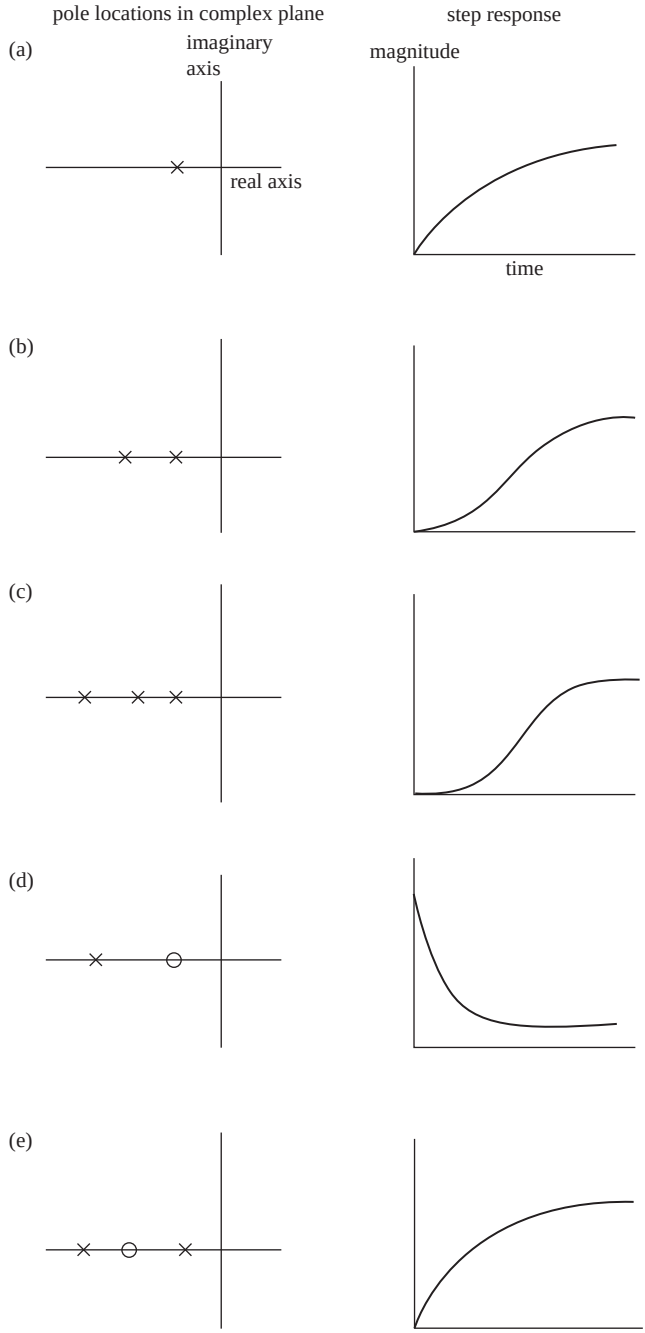


Figure 4.8 Examples of pole–zero diagrams and their associated step responses

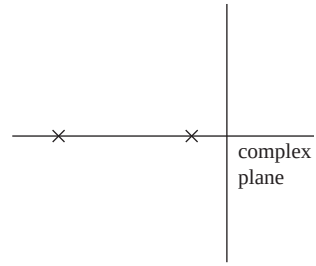


Figure 4.9 Presumed initial position of system poles

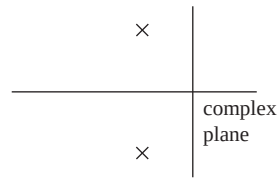


Figure 4.10 The required position of the system poles

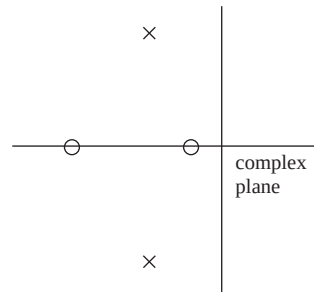


Figure 4.11 Poles and zeros of a synthesised system (controller) that when connected in series with G will 'move' the poles to the required positions

system by an element *D* having pole–zero diagram Figure 4.11 will cancel the poles of *G* and produce the required poles. This technique is called pole placement.

Notice carefully that the unwanted poles of *G* are not removed – rather their effect on the external behaviour is cancelled out by the zeros of *D*.

Two difficulties can arise when pole cancellation is used.

(i) Cancellation may not be exact, or, if initially exact, may not remain so. This is particularly important where the poles whose cancellation is intended are unstable poles.

(ii) A system in which poles have been cancelled out by coincident zeros only appears to have a simple form. Internally, the structure representing the cancelled terms is still present although it does not affect, nor can it be affected by, outside events.

The redundant internal structure leads to difficulties and anomalies, particularly in those cases where matrix techniques are to be applied. This topic is discussed again in Sections 8.10, and 8.11.1.

4.10 Moving the poles of a closed loop system to desirable locations: the root locus technique

Consider the transfer function system (Figure 4.12):

$$G(s) = \frac{C}{(s+1)(s+3)}$$

which has poles at $s = -1$, $s = -3$. If the same system is connected in a closed loop (Figure 4.13), then, as shown in Section 4.5, the overall transfer function for the configuration is

$$\frac{G(s)}{1+G(s)} = \frac{C}{(s+1)(s+3)} \Bigg/ \left(1 + \frac{C}{(s+1)(s+3)}\right) = \frac{C}{(s+1)(s+3)+C}$$

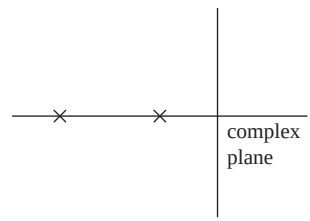


Figure 4.12 Poles of G(s) = C/[(s + 1)(s + 3)]

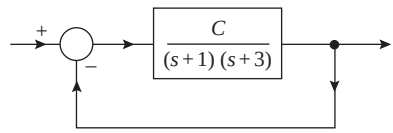


Figure 4.13 G(s) connected into closed loop

The poles of the closed loop configuration are found by equating the denominator of the transfer function to zero. In this case, the equation to be solved is

$$(s+1)(s+3) + C = 0$$

The solutions are $s = -2 \pm \sqrt{1-C}$

For $C < 1$, the poles are real, unequal
$C = 1$, the poles are real, equal
$C > 1$, the poles are complex conjugates

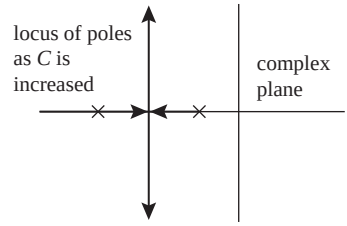


Figure 4.14 A root locus diagram for G(s), showing how the closed loop poles move with increasing values of C

A diagram (Figure 4.14) showing how the poles move with changing C is called a root locus diagram. With the aid of the root locus diagram, we can decide on the value of C that will result in the closed loop poles being in desirable positions in the complex plane. Chestnut and Mayer (1959), Chapter 13, has many examples of root locus configurations. More recent references, such as Dorf *et al.* (2011), do not go into such detail but will be adequate for many purposes.

4.11 Obtaining the transfer function of a process from either a frequency response curve or a transient response curve

A frequency response curve is a curve that illustrates how a system's steady state response to sinusoidal signals varies as a function of the frequency of those signals (frequency response is discussed in Chapter 5).

A transient response curve is a curve that records a system's behaviour as a function of time immediately after the application of a stimulus to the system.

A non-minimum phase system is a system whose transfer function has one or more zeros in the right half complex plane (the reasons for the name and some discussion can be found in Chapter 8).

Experimental tests may produce frequency response curves or transient responses and these may need conversion to transfer functions to start design in the pole–zero domain. (Truxal (1955), p. 345 *et seq*., has a masterly and detailed treatment of these topics – highly recommended.)

4.11.1 Obtaining a transfer function from a given frequency response curve

The subject of filter synthesis tackles the problem in great detail (Guillemin, 1957). However, for control purposes, the problem is simpler and, in particular, a transfer function that has the desired magnitude response is likely also to have the desired phase angle characteristics. (In fact, for minimum phase transfer functions, the phase characteristic is completely determined by the gain characteristic (see H.W. Bode cited in Truxal, 1955, p. 346).)

Thus, if the magnitude characteristic can be approximated by straight line segments, then an approximate transfer function may be quickly produced using (inversely) the rules for straight line sketching of Bode diagrams (Dorf *et al.*, 2011).

4.11.2 Obtaining a transfer function from a transient response curve

Let the test signal be $u(t)$ and the resulting transient response be $y(t)$, then the transfer function G may be determined using the expression

$$G(jw) = \frac{\mathfrak{F}\{y(t)\}}{\mathfrak{F}\{u(t)\}}$$

where $\mathfrak{F}$ indicates Fourier transformation.

In the days of 'hand computation', ingenious methods were devised to approximate the necessary Fourier transformation. Some of these methods are still of interest since they give insight into how the shape of a transient curve actually carries the transfer function information. For instance, Guillemin's technique (see Truxal (1955), p. 379) involves approximation of the transient response by segments of polynomials, followed by repeated differentiation, resulting in a finite set of impulses from which the transfer function is written by inspection.

4C Convolution: what it is

Let the system of transfer function $G(s)$ have the response $g(t)$ to a unit impulse (Figure 4.15). The response to any other sort of input can then be visualised as the response to a train of impulses that approximates the function (Figure 4.16).

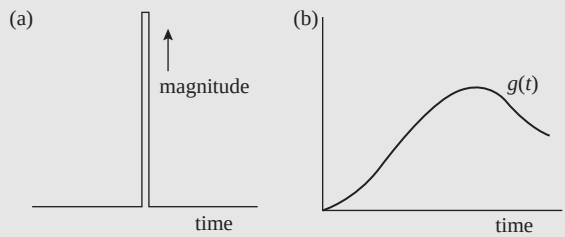


Figure 4.15 (a) A unit impulse at l = 0; (b) the response G(t) of a system to a unit impulse

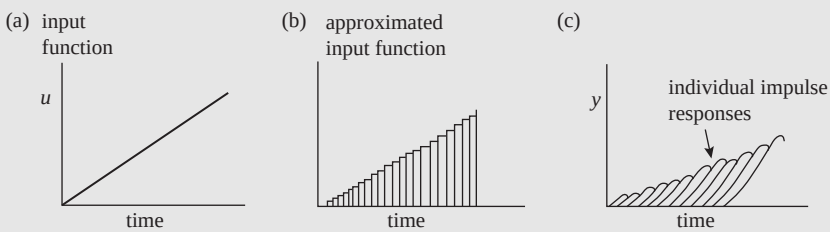


Figure 4.16 (a) A ramp input; (b) a ramp input approximated by impulses; (c) the response of a system to the individual impulses of (b)

Any one of the individual impulse response curves in Figure 4.16(c) can be expressed as $u(\tau)g(t-\tau)$, where τ is the time of application of the impulse. Linearity allows us to say that

$$y(t) = \int_0^t u(\tau)g(t-\tau)\,d\tau$$

and this expression, unpopular amongst students, is called the *convolution integral*.

We can avoid convolution or, more correctly, allow the Laplace transform to take care of it, as follows:

Let $y(t) = u(t) * g(t)$

Where $*$ indicates convolution.

Then, by the properties of Laplace transforms

$$y(s) = u(s)G(s)$$

and

$$y(t) = \mathcal{L}^{-1}\{u(s)G(s)\}$$

In other words, transform multiplication corresponds to convolution of time functions.

To complete the discussion, we illustrate the use of the transform method to calculate the response of a system to a stimulus.

Let the system have the impulse response $g(t) = \exp(-t)$ (this implies $G(s) = 1/(s+1)$), and assume the input u is a ramp function, i.e. $u(t) = t$, implying $u(s) = 1/s^2$. Then

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s+1}\right\}$$

(obtained by the use of partial fractions). Finally, inversion produces

$$y(t) = t - 1 + e^{-t}$$

4.12 Determination of transfer functions by cross-correlation

The cross-correlation of functions $u(t)$ and $y(t)$ is given by

$$R_{uy}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} u(t-\tau)y(t)\,dt$$

If u is the input and y the corresponding output of a system G, then

$$y(t) = \int_{-\infty}^{\infty} g(\tau)u(t-\tau)\,d\tau$$

And after combination of the two expressions and some manipulation, we obtain

$$R_{uy}(\tau) = \int_{-\infty}^{\infty} g(x) R_{uu}(\tau - x)\, dx$$

where R_{uu} is the autocorrelation function of the signal u.

Under the special condition that the signal $u(t)$ is white noise, whose autocorrelation function is an impulse, the cross-correlation function $R_{uy}(\tau)$ is the system's impulse response and the Fourier transform or Laplace transform of this function is the system's transfer function.

4D Calculation of resonant frequencies from the pole–zero diagram

System responses can be calculated from the pole–zero diagram using approaches that are well described in, for example, Maddock (1982) or D'Azzo (2003). These approaches are not really competitive as numerical algorithms but they can be very instructive. Thus, Figure 4.17 has been drawn to illustrate resonance in a second-order system – resonance occurs when the product ab of the lengths a, b in the figure is a minimum as the arbitrary point p on the vertical axis (representing frequency) is varied. The calculation for the minimum value is carried out beneath the diagram, resulting in a formula for the resonant frequency.

We define

$$\boldsymbol{J} = \left(r^2 + (h-w)^2\right)\left(r^2 + (h+w)^2\right)$$

Resonance occurs when ab is minimum, i.e. when $(ab)^2 = \boldsymbol{J}$ is minimum:

$$\begin{aligned}\frac{d\boldsymbol{J}}{dw} &= (r^2 + h^2 - 2hw + rw^2)(2h + 2w) \\ &\quad + (-2h + 2w)(r^2 + h^2 + 2hw + w^2) \\ &= r^2 + w^2 - h^2\end{aligned}$$

Thus, the resonant frequency ω_r must satisfy

$$\omega_r^2 = h^2 - r^2$$
$$\omega_r = \left(h^2 - r^2\right)^{1/2}$$

Now, damping factor ζ satisfies

$r = \omega_n \zeta$ (since ω_n is a vector from origin to pole)

$$h = \omega_n \sqrt{1 - \zeta^2}$$

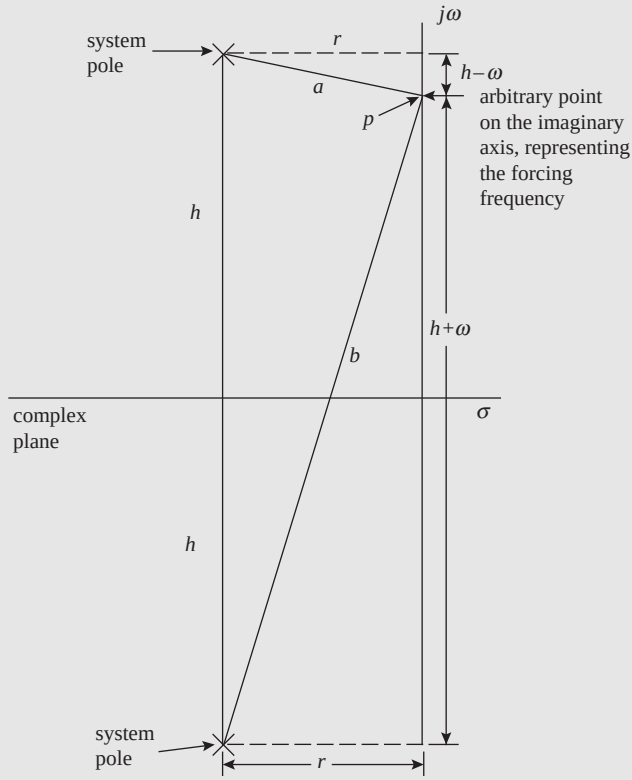


Figure 4.17 Construction in the complex plane for the graphical determination of resonant frequency

hence,

$$\omega_r^2 = -\omega_n^2\zeta^2 + \omega_n^2(1 - \zeta^2)$$

from which

$$\omega_r = \omega_n\sqrt{1 - 2\zeta^2}$$

4E Derivation of a formula for damped natural frequency

Following the application of a step input, the output of a stable system having a pair of complex poles oscillates at a frequency ω_d within a decaying exponential envelope, where ω_d is called the damped natural frequency.

Let p be a vector from the origin of the complex plane to one of the system poles (Figure 4.18).

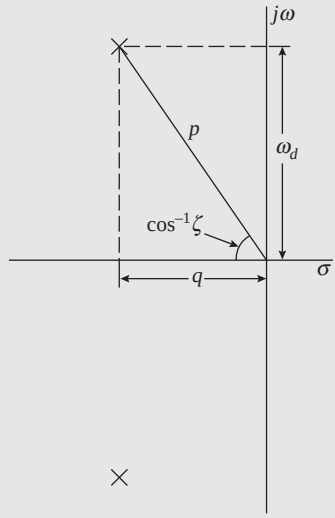


Figure 4.18 Construction in the complex plane for the determination of damped natural frequency

Undamped natural frequency ω_n is numerically equal to the length of the vector p.

Damping factor ζ is the cosine of the angle that the vector p makes with the negative real axis.

Damped natural frequency is given by the length of the projection of the vector p onto the imaginary axis.

Then referring to Figure 4.18,

$$p^2 = q^2 + \omega_d^2$$

and

$$q = \omega_n \zeta, \;\; p = \omega_n$$

therefore

$$\omega_d^2 = \omega_n^2 - \omega_n^2 \zeta^2$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

4F The root locus of a system with open loop poles and zeros located as in Figure 4.19 will include a circle centred on the zero

The closed loop transfer function of the system shown in Figure 4.19 is

$$\frac{C(s+3)}{(s+1)(s+2)+C(s+3)}$$

or (for ease of manipulation), putting $s=p-3$, to move the origin to the point $s=-3$.

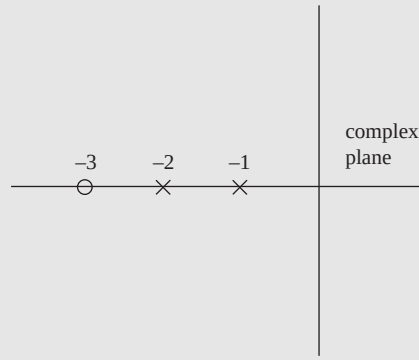


Figure 4.19 The pole–zero diagram of the second-order system under study in this section

The characteristic equation is

$$p^2+(C-3)p+2=0$$

This is the equation of a circle, with centre at $(-3, 0)$ and radius 2. To appreciate this, solve the characteristic equation, obtaining

$$\mathrm{R}(p)=\frac{3-c}{2}, \quad \mathrm{I}(p)=\sqrt{2-\left(\frac{3-C}{2}\right)^2}$$

$$\sqrt{\mathrm{R}(p)^2+\mathrm{I}(p)^2}=\sqrt{2}$$

Here R, I denote real and imaginary part, respectively.

Note: Source material and suggestions for further reading to support the topics of this chapter will be found in Chapter 18.

Chapter 5

Frequency response methods

5.1 Introduction

Frequency response methods have a physical explanation that is readily understandable without any mathematics. In addition, the methods are design oriented, link easily between practical results and differential equation methods, and have been proven to work well in many practical design situations.

The 'home territory' for frequency response methods has traditionally been in servomechanism, process control and aerospace applications, and they have been rather resistant to applications outside these areas.

5.2 Design using frequency response methods: initial explanation

Frequency response methods have a distinguished history with Harold Nyquist (1932) and Hendrik Bode (1945) being credited with early fundamental work that remains relevant.

Control design in the frequency domain involves the following basic ideas:

(i) The performance of a system H that is to be synthesised may be approximately characterised by its bandwidth, i.e. by the range of frequencies to which it will respond.
(ii) The bandwidth of any process G that is to be controlled may be measured experimentally or calculated analytically by straightforward means.
(iii) The necessary frequency characteristics of a controller D may be determined graphically from information on G and H, such that the performance in (i) is obtained.
(iv) Sufficient stability of the resulting control loop is easily taken care of as part of the design method.

5.3 Frequency response of a linear system

A linear dynamic system consists mathematically of the following (repeated) operations: multiplication by a constant, differentiation, integration and summation,

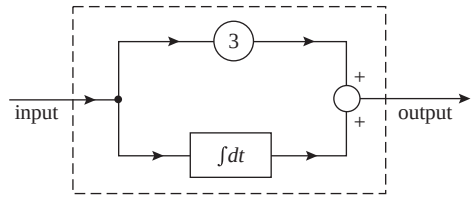


Figure 5.1 A linear system consisting of a gain and integrator

and of no other types of operation. Therefore, the response of a linear system to a sinusoid must necessarily be also sinusoidal – wave shape and frequency both being invariant under linear transformation.

Illustration: A linear system has the configuration shown in Figure 5.1. The input to the system is multiplied by a gain of 3 in the upper arm. It is integrated in the lower arm and the two signals are added to become the output. Thus, if the input is a sinusoid of unit amplitude and frequency 1/4 rad/s (i.e. the input is the signal $\sin t/4$), then the output will be

$$\begin{aligned} 3\sin\frac{t}{4}+\int \sin\frac{t}{4}dt &= 3\sin\frac{t}{4}-4\cos\frac{t}{4} \\ &= 5\left(\frac{3}{5}\sin\frac{t}{4}-\frac{4}{5}\cos\frac{t}{4}\right) \\ &= 5\left(\cos\alpha\sin\frac{t}{4}-\sin\alpha\cos\frac{t}{4}\right) \\ &= 5\sin\left(\frac{t}{4}-\alpha\right) \end{aligned} \tag{5.1}$$

where $\alpha = \cos^{-1}\frac{3}{5}$, and we confirm that the signal remains sinusoidal of the original frequency, but the amplitude has changed and there is a phase shift α between input and output sinusoids.

By the *frequency response* of a system we mean a table or graph showing the output amplitude and phase difference as a function of frequency when a sinusoid of unit amplitude is applied to the system (it being assumed that all transient effects have died away before output measurements are taken).

5.4 The Bode diagram

The Bode diagram allows frequency response information to be displayed graphically. The diagram (Figure 5.2) consists of two plots, magnitude and phase angle, both against frequency on the horizontal axis.

5.5 Frequency response and stability: an important idea

If, for some particular frequency ω, the block G has unity gain and –180° phase shift, then the closed loop system shown in Figure 5.3 will be in continuous oscillation at frequency ω.

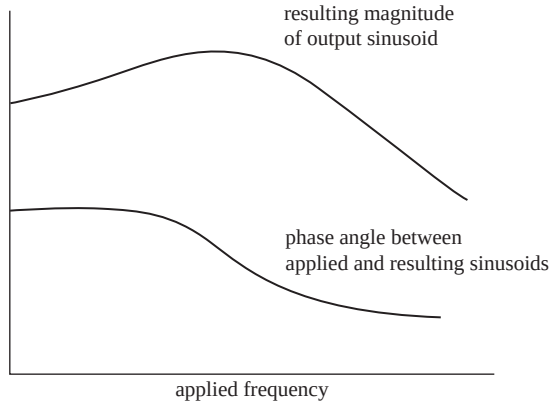


Figure 5.2 The form of a Bode diagram

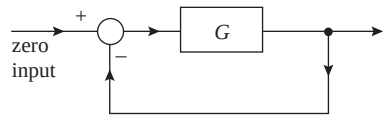


Figure 5.3 A block of transfer function G *with unity feedback*

Explanation: A sinusoid of frequency ω, once input to the block G, will be subjected to two phase shifts of 180° (one at G and one at the comparator (multiplication by -1 and phase-shifting by 180° having the same effect on a continuous sinusoid)) and will pass repeatedly around the loop without attenuation, since the loop gain at frequency ω is unity.

In practice, special log-linear axes are used for Bode diagrams with frequency on a logarithmic scale and magnitude not plotted directly but only after conversion to decibels (dB). Under these special circumstances, the Bode plots for magnitude for most simple transfer functions can be approximated by straight line segments. In the logarithmic domain, products of transfer functions are replaced by summations of individual logarithmic approximations. Hence, the Bode diagram magnitude characteristic for a moderately complex transfer function can easily be produced by summing a few straight line approximations.

The Bode diagram's popularity derives from the ease with which it may be sketched, starting from a transfer function; the ease with which it may be obtained by plotting experimental results; and from its usefulness as a design tool.

Implication: For stability of the closed loop system shown in Figure 5.3, at that frequency where the phase shift produced by G is –180°, the loop gain must be less than unity. Notice that the stability of the complete closed loop is being inferred from frequency response information referring to the block G alone.

5.6 Simple example of the use of the foregoing idea in feedback loop design

Block G of Figure 5.4(a) has the frequency response shown graphically in Figure 5.4(b). Choose the largest numerical value for the gain C, consistent with stability of the loop of Figure 5.4(c).

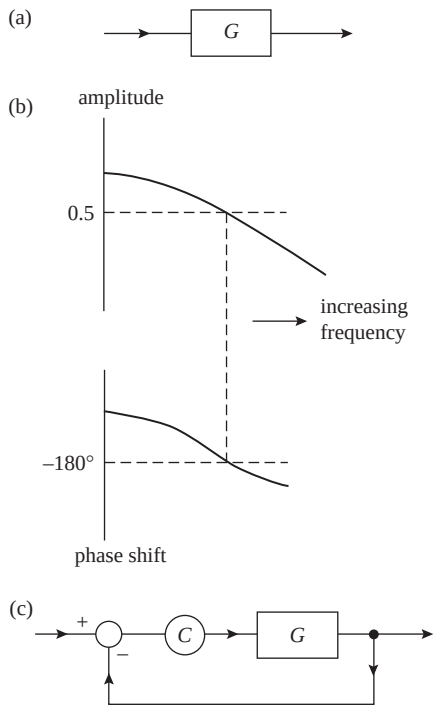


Figure 5.4 (a) A block of transfer function G; (b) the frequency response of G; (c) the gain C in the loop is to be set to the highest possible value, consistent with stability of the loop

At the frequency where the phase shift of block G is –180°, the gain of G is 0.5, i.e. G multiplies sinusoids by a factor of 0.5 at that frequency. Thus, it is clear that the gain C could be set to $C = 2$ to bring the system to the stability limit. (The gain C affects only amplitude – it has no effect on the phase shift curve.)

5.7 Practical point: the need for stability margins

The gain C cannot in practice be set to the stability limit – rather C must be set so that a *stability margin* is observed. This ensures that, even allowing for the inevitable variations in all real systems, stability will still be obtained. Further, the type of response to inputs other than sinusoids will then not be too oscillatory, as would be the case were the loop gain set at the stability limit.

5.8 General idea of control design using frequency response methods

Control design in the frequency domain is quite a specialist subject, requiring considerable experience and detailed knowledge. However, in principle, what is involved is, in addition to the original process, a *compensator D* and, as before, a gain *C*, to be chosen (Figure 5.5).

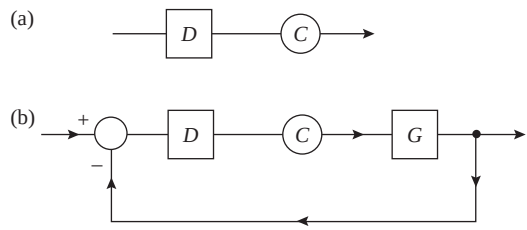


Figure 5.5 (a) A compensator D in series with a gain C; (b) the combination of (a) in position to control the process G

Treating *GD* as a pseudo-process, the choice of gain is made exactly as before. By suitable choice of the compensator *D*, systems satisfying particular specifications can be built up. In particular, systems with a flat frequency response up to a given frequency may be specified. Alternatively, undesirable resonance peaks in the frequency response for *G* may be cancelled out by proper choice of *D*.

Suppose that *G* is an existing process, like an electromechanical rotating device whose position is to be controlled.

D is a controller, to be designed, which can contain frequency-sensitive elements. *C* is, as before, a simple numerical gain.

The problem: Design *D* and choose *C* to obtain a closed loop system having high bandwidth. The frequency response of the block *G* is supposedly known (it has been measured or calculated).

Procedure: Design *D* so that *G* and *D*, taken together, have a phase characteristic that reaches –180° at a much higher frequency than was the case for *G* alone, then choose the gain *C* so that the necessary stability margin is obtained.

In principle: A controller (or compensator) *D* is being used to modify the phase characteristics of *G* in such a way that a high gain *C* can be used without incurring stability problems. Such a high loop gain brings the high loop bandwidth desired by the designer.

5.9 Obtaining the frequency response of a system experimentally

A frequency response analyser makes the work easy since this device generates the necessary sinusoids, measures the responses and produces digital displays and plots of amplitudes and phase angles.

The analyser injects a pure sinusoid (actually, in commercial analysers, both sine and cosine waves are usually injected simultaneously for noise cancellation purposes) at one particular frequency ω within this range for an integral number of cycles N, meaning that the test at this frequency ω must last for T seconds, where $T = N\pi/\omega$ seconds and N is an integer.

Let the input to the system be ($u = a \sin(\omega t)$), then under the assumptions of linearity implied in this test, the output of the system being tested must be ($y = b \sin(\omega t + \phi)$). Averaging over a sufficient number of cycles to yield reliable results, the analyser produces estimates of $b/a = |G(j\omega)|$, which is the system gain at frequency ω, and $\phi = \angle G(j\omega)$, which is the phase angle between input and output at frequency ω; $(b/a; \phi)$ is just one point on the frequency response diagram but the analyser sweeps automatically over the selected frequency range to produce a complete data set.

Note: It can be shown that averaging over a large integral number of cycles significantly reduces the effects of noise and non-linearity, which in the long term do not correlate with the pure sine wave input. See Wellstead (2003) for coverage of all the above aspects.

5.9.1 *Obtaining the frequency response of a system experimentally: some practical difficulties*

Even the very simplest of systems can prove surprisingly difficult to identify in practice. That is just one of several reasons for the much-discussed gap between the promises of theory and the achievements of practice – if a meaningful model can't be obtained, a meaningful control design is necessarily difficult to produce.

Some of the difficulties frequently encountered in practice are as follows:

- It often proves quite a challenge to connect the sinusoidal stimulus signal $u(t)$ from the analyser to the system to be identified. Few real processes have a pair of convenient terminals that allow this; many industrial interfaces use mark-space chopping of power to avoid the cost of continuously variable power amplifiers; unwieldy devices need to be rigged up to produce linearly reciprocating sine waves to excite some types of mechanical systems.
- Industrial processes are often already operating in 'some sort of closed loop arrangement' and it is not possible to isolate such processes for testing.
- The product is often a key part of the process that is to be tested. Industrial processes, in many cases, cannot be considered to exist separately from the product being produced – managers may not take kindly to sinusoidal variations being induced into the products.
- Testing takes a very long time if low frequencies are involved. This applies particularly to large processes that tend to operate in the low-frequency end of the spectrum.
- Electromechanical systems tend to move in a series of jerks when confronted with very low frequency signals. They tend to move erratically, giving inconsistent results, for high frequencies. Both effects can be attributed to the presence of non-linearities. Usually stiction is the cause of the low-frequency

jerking phenomenon, whereas backlash in mechanisms is the source of most high-frequency erratic behaviour. (At high frequencies, attenuation is severe, drive signals are of small amplitude and backlash becomes significant.)

- Systems whose output has a non-zero mean level (especially a mean level that follows a long-term large amplitude ramp) are very difficult to deal with.

This daunting list should not be taken to imply that frequency response testing can never be applied successfully in practice! However, it is true that only a somewhat limited class of processes can be successfully tested. Many of these are in the aerospace field. For industrial processes, other approaches are often used.

5.10 Design based on knowledge of the response of a system to a unit step input

When an input signal of the form shown in Figure 5.4(a) is applied to a system, the resulting response is called the *unit step response of the system* (Figure 5.6). It can be shown that all the information contained in a system's frequency response is also contained in the system's step response. However, the following points should be noticed:

(i) The step response of a process is very much easier to obtain than the frequency response (in some cases just switch it on!). Even industrial processes on which experimentation is forbidden can be persuaded to yield step response information.

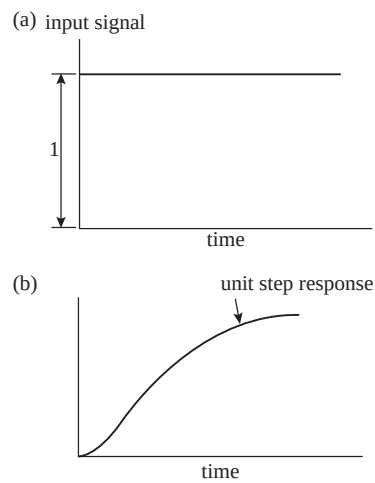


Figure 5.6 (a) The input to a system; (b) the output of the system in response to the input (a) is called the unit step response of the system

(ii) No very attractive design methods exist that use the step response as their input. However, the semi-empirical Ziegler–Nichols methods (one of which is based around an experimentally obtained step response) exist to allow the rapid tuning of coefficients in three-term controllers. Three-term controllers are the highly successful no-nonsense limited-ability devices that actually control a very high percentage of real industrial processes. See Section 9.3 for further information.

(iii) Computer packages can very easily transform a system's step response into an equivalent frequency response. Thus, the easy-to-obtain step response can serve as an input to frequency-response-based design approaches. However, if such an approach is used, it is recommended to obtain several step responses corresponding to different input amplitude changes and to repeat these, starting from different levels, for negative going as well as for positive going input steps to ensure that asymmetry and non-linearity are discovered so that, if severe, these effects may be compensated for.

(iv) Many attractive published methods of system identification turn out to be inapplicable to real industrial situations because they require the process to be available for experimentation or they neglect other significant realities. This means that step responses, which can almost always be obtained even during normal process operation, will often be the only experimental data available to the modeller. A common sense approach would therefore be to approximate nothing and to search for that process model (or set of models) which gives the best time-domain fit to actual unsmoothed recorded step responses from the process. One monograph that does pursue such an approach (using orthonormal Laguerre functions for the fitting) is by Wang and Cluett (2000).

5.11 How frequency response is obtained by calculation from a differential equation

Suppose that a system is represented by the differential equation

$$\frac{dy}{dt} + ay = u \tag{5.2}$$

and that the input u is a sinusoidal signal $u = \delta \sin \omega t$. It is not difficult to solve the equation

$$\frac{dy}{dt} + ay = \delta \sin \omega t \tag{5.3}$$

using straightforward integration or Laplace transforms. For frequency response purposes, the transient part of the solution is not usually of interest and only the

particular integral, describing the periodic behaviour, needs to be considered. Using the operator D method for this we obtain:

$$
\begin{aligned}
(D+a)y &= \delta \sin \omega t \\
y &= \frac{\delta}{D+a} \sin \omega t = \frac{\delta(D-a)}{D^2-a^2}\sin\omega t \\
&= \frac{\delta(D-a)}{-\omega^2-a^2}\sin\omega t = \frac{-\delta(D-a)}{\omega^2+a^2}\sin\omega t \\
&= \delta\left(\frac{a\sin\omega t - \omega\cos\omega t}{\omega^2+a^2}\right) \\
&= \frac{\delta}{\sqrt{\omega^2+a^2}}\left(\frac{a\sin\omega t - \omega\cos\omega t}{\sqrt{\omega^2+a^2}}\right) \\
&= \frac{\delta}{\sqrt{\omega^2+a^2}}(\cos\alpha\sin\omega t - \sin\alpha\cos\omega t)
\end{aligned}
$$

where $\alpha = \tan^{-1}(\omega/a)$

$$
\begin{aligned}
&= \frac{\delta}{\sqrt{\omega^2+a^2}}\sin(\omega t - \alpha) \\
&= m\sin(\omega t + \phi)(\text{say})
\end{aligned}
\tag{5.4}
$$

Thus,

$$m = \frac{\text{Magnitude of output sinusoid}}{\text{Magnitude of input sinusoid}} = \frac{1}{\sqrt{\omega^2+a^2}}$$

ϕ = Phase difference between input and output sinusoid $= -\tan^{-1}(\omega/a)$

If we return to the transfer function of the original system,

$$G(s) = \frac{1}{s+a}$$

and obtain

$$G(j\omega) = \frac{1}{j\omega + a}$$

then we find that the magnitude m is the same thing as the modulus of the complex number $G(j\omega)$ while the phase angle ϕ is the argument *of* $G(j\omega)$. In other words, if $G(j\omega)$ is expressed in $R\angle\alpha$ form, then $R = m$ and $\alpha = \phi$. These relations allow the frequency response of a transfer function to be calculated very simply by determination of the modulus and argument of a complex number as a function of frequency – there is no requirement (since these relations are available) to solve differential equations.

It can easily be demonstrated by simple examples that the substitution method, as just described, gives the same results as (5.4). The formal justification for setting $s = j\omega$ to obtain frequency response information from the transfer function can be as follows:

Let a process of transfer function $G(s)$ and impulse response $g(t)$ receive as input the complex sinusoid $\exp(j\omega t)$. Then the steady state response y_{ss} can be found by convolution to be

$$\begin{aligned} y_{ss} &= \int_0^\infty g(\tau)\exp(j\omega(t-\tau))d\tau \\ &= \exp(j\omega t)\int_0^\infty g(\tau)\exp(-j\omega t)d\tau \end{aligned} \tag{5.5}$$

Comparing the term under the integral sign with the defining equation for $G(s)$:

$$G(s) = \int_0^\infty g(\tau)\exp(-s\tau)d\tau \tag{5.6}$$

we see that

$$y_{ss} = \exp(j\omega t)G(j\omega) \tag{5.7}$$

i.e. the output is also the complex sinusoid of frequency ω but of magnitude $|G(j\omega)|$ and with phase difference (compared with the input) of $<G(j\omega)$.

5.12 Frequency response testing can give a good estimate of a system's transfer function

Assume that frequency response testing has produced the magnitude curve of Figure 5.7. Then it is clear by inspection that the system can be modelled by a transfer function of the form

$$G(s) = \frac{C}{(1+sT_1)(1+sT_2)}$$

where $T_1 = 1/\omega_1$, $T_2 = 1/\omega_2$, $C = 10^{7/20}$ (to see this, sketch the form of the Bode plot for the given $G(s)$).

Questions to be asked about frequency response testing:

(i) On what proportion of real systems can meaningful frequency response tests be carried out?
(ii) What proportion of successfully completed frequency response tests lead to an easily interpreted set of data?
(iii) How often can a real control system be designed using an experimentally obtained frequency response model?

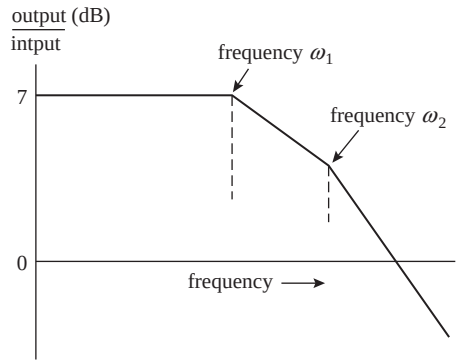


Figure 5.7 The supposed frequency response (magnitude curve) of an unknown system

(iv) Overall, roughly what proportion of real control systems are actually designed via these routes?

5.13 Frequency response of a second-order system

A first-order system is abnormally simple. The step response is exponential. The frequency response (magnitude) plot decays (or for an unstable system, increases) monotonically. Oscillation and resonance are not possible.

A second-order system, although structurally simple, can in many ways be considered as a reliable outline approximation for a whole class of systems of higher order. For instance, when trying to visualise a concept, it will often be sufficient to think of dynamic effects in terms of their second-order approximation. For the reasons just given, it is very useful to understand the frequency response of a normalised second-order system.

Every second-order (linear) system can be converted into the standard form

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \omega_n^2 u$$

with transfer function

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

putting $s = j\omega$

$$\begin{aligned} G(j\omega) &= \frac{\omega_n^2}{\omega_n^2 - \omega^2 + j2\zeta\omega_n\omega} \\ &= \frac{1}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2 + j2\zeta\left(\dfrac{\omega}{\omega_n}\right)} \end{aligned}$$

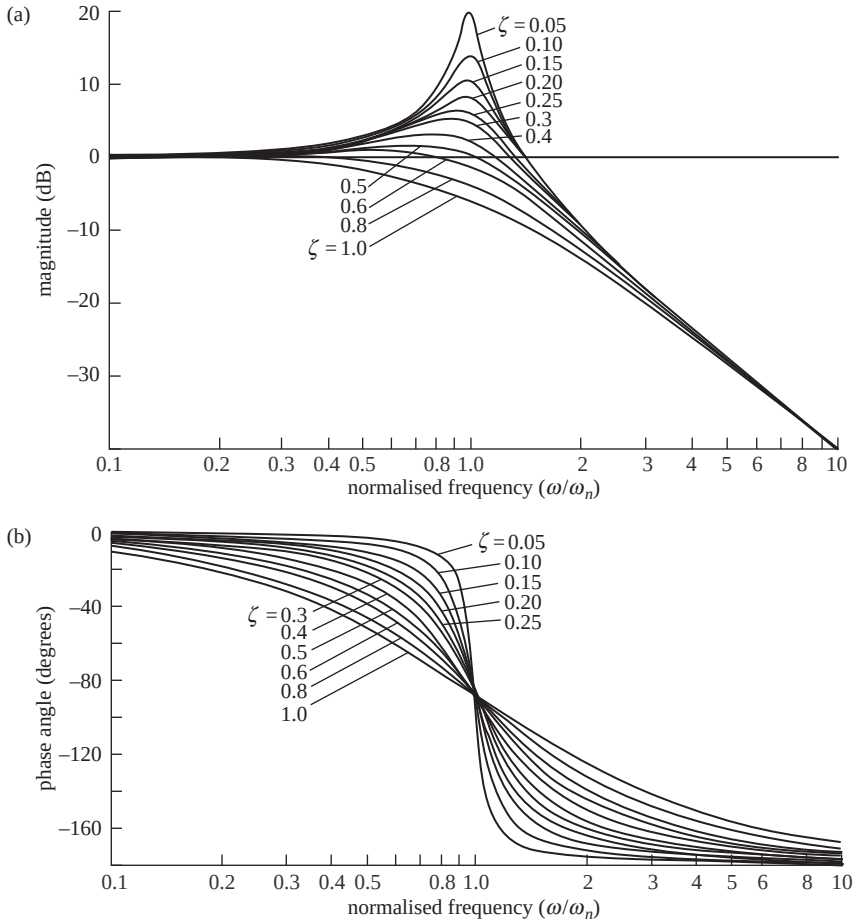


Figure 5.8 Frequency response for a second-order system with different damping factors ζ: (a) magnitude curve; (b) phase curve

We can obtain universally useful Bode diagrams of the plot against 'dimensionless frequency' ω/ω_n. Such plots are shown in Figures 5.8(a) and (b).

5A The frequency response of a system with poles and/or zeros near to the imaginary axis

A system has the poles and zeros shown in Figure 5.9(a). As the applied frequency moves up the imaginary axis there will be a notch in the magnitude response as the zero is passed and a peak as the pole is passed. The magnitude plot of frequency response will have the approximate form of Figure 5.9(b) (see Dorf *et al.* (2011) for further background).

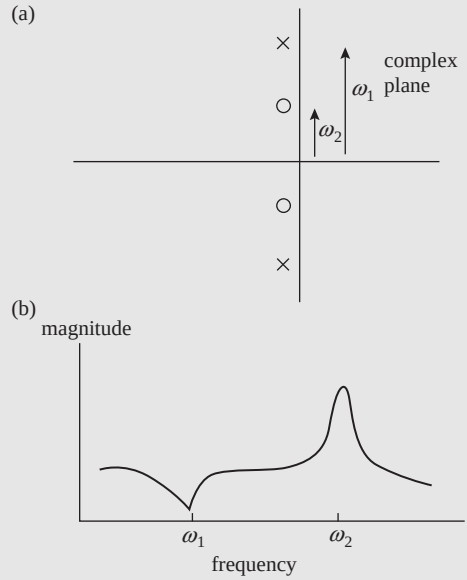


Figure 5.9 (a) A pole–zero diagram in which the poles and zeros are close to the imaginary axis; (b) the form of the (magnitude) frequency response corresponding with (a)

5B Some interesting and useful ideas that were originated by Bode

Bode (1945) showed that (provided non-minimum phase systems are excluded) the magnitude and phase characteristics are totally interdependent. That is to say, given a magnitude characteristic for a Bode diagram, the phase characteristic is completely determined, and conversely. The following is based directly on Chestnut and Mayer (1959), which should be consulted for additional detail.

Bode's theorem 1 states, retaining his original notation: the phase shift of a network or system at any desired frequency can be determined from the slope of its attenuation/frequency characteristic over the range of frequencies from $-\infty$ to $+\infty$. The slope of the attenuation/frequency characteristic at the desired frequency is weighted most heavily, and the attenuation/frequency slope at frequencies further removed from the desired frequency has lesser importance:

$$B(\omega_d) = \frac{\pi}{2}\left|\frac{dA}{du}\right|_0 + \frac{1}{\pi}\int_{-\infty}^{+\infty}\left[\left|\frac{dA}{du}\right| - \left|\frac{dA}{du}\right|_0\right]\ln\coth\left|\frac{u}{2}\right|du \qquad (5.8)$$

where $B(\omega_d)$ = the phase shift of the network in radians at the desired frequency ω_d

A = attenuation in nepers where 1 neper = $\ln|e|$ (*Note*: 1 neper equals about 8.7 decibels.)

This curve provides a valuable insight into the relation between magnitude and phase characteristics (Figure 5.10).

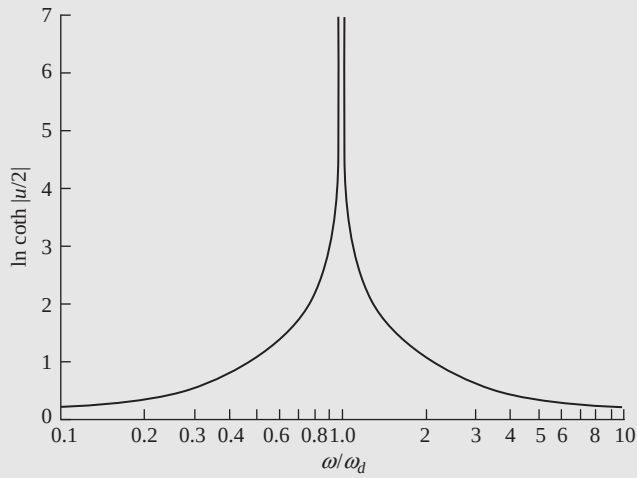


Figure 5.10 Weighting function for use with (5.8), where $u = \ln \omega/\omega_d$

In most situations, the phase shift is determined largely by the first term of (5.8). From this point of view, it appears that, for the phase shift to be less negative than –180° at frequencies in the vicinity of the $-1 + j0$ point, the attenuation slope should be less than 2 nepers per unit of u or less than 40 dB per decade over a fairly broad range of frequencies.

The following simple and very useful rule (again due to Bode and verifiable from the material given above) allows (minimum phase) stable systems to be synthesised using only the magnitude plot.

'A system will be stable if the slope of the Bode magnitude plot in the region of 0 dB is –20 dB/decade and if this slope is maintained for a region of ±0.5 decade about the 0 dB crossing point'.

This simple rule is only approximate and it is indeed rather conservative. However, it is a very useful rule for making a first cut design (Truxal, 1955, p. 46).

5.14 Nyquist diagram and Nichols chart

The information in a Bode diagram may be represented in alternative forms. Representation in polar co-ordinates results in the *Nyquist diagram* – this is a locus in the complex plane with frequency being a parameter on the locus.

The *Nichols chart* is a plot of magnitude against phase angle. This diagram is again a locus along which frequency appears as a parameter. The Nichols chart is used with a special overlay that assists control design.

The Bode diagram, Nyquist diagram and Nichol's chart are complementary techniques in the armoury of the frequency-response-oriented system designer. There is a very extensive literature.

Note: Source material and suggestions for further reading to support the topics of this chapter will be found in Chapter 16.

Chapter 6

Mathematical modelling

6.1 Approaches to mathematical modelling

Figure 6.1 shows a general situation that is to be modelled. External influences (controls, raw material characteristics, environmental influences and disturbances) are contained in vector u. Available information (measurements, observations, other data) are contained in vector y. The vector x contains internal variables fundamental to the situation. This vector may be of no interest whatever, except as a building block to the modeller. Alternatively, x may be of great interest in its own right. We assume that there are available data sets $\{u_i, y_i\}$ for the modeller to work on.

Approach (1) is to fit numerically a dynamic linear input–output model G_i to each data set $\{u_i, y_i\}$. This is very easy, but:

(i) G_i may not fit the data well for any i. Such an effect may be encountered when the situation is non-linear and/or time varying.
(ii) Different data sets $\{u_j, y_j\}$, $\{u_k, y_k\}$ that are supposed to arise from the same mechanism may give rise to widely differing models G_j, G_k.
(iii) Non-standard types of information, contained within the vectors u_i, y_i may be impossible to accommodate within a standard identification procedure.

Approach (2) is to construct a set of interlinked physically inspired equations, involving the vector x, that approximate (possibly grossly) the mechanisms that are thought to hold in the real process. The data sets $\{u_i, y_i\}$ are then used quantitatively to fix numerical values for any situation-specific coefficients and, when best values have been found, to verify the performance of the resulting model.

Approach (3) is to fit an empirical black box model, typically a neural network, to as wide a range of input–output data as possible in the hope of obtaining

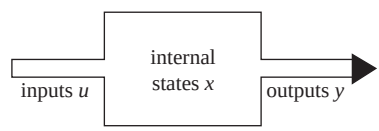


Figure 6.1 A general situation that is to be modelled

a single non-linear relation that represents all the cases presented. The expectation is that the behaviour of the model so obtained will generalise sufficiently well to act as a useful model of the process. Neural net modelling is discussed in Section 16.2.

6.2 Methods for the development of mathematical models

Although control theory is a fairly coherent well-defined body of concepts and knowledge, supported by techniques, the activity of mathematical modelling is ill-defined and its practitioners are scattered among many disciplines. Thus, in science, models are often used to explain phenomena as, for instance, the Bohr model of the atom or the wave theory of electromagnetic propagation. Such models are essentially visualisations of mechanisms. Far removed from this are those models, usually implicit and sometimes fictitious, by which politicians claim to predict future rates of employment or inflation.

We can propose that the science models contain – and this is their fundamental characteristic – a representation of physical variables. The second group may be, in the extreme, no more than extrapolations of past trends. Constructing a model in the first category is primarily a matter of bringing together, combining and refining concepts to produce an object called a model (usually it will consist of a set of equations).

A key question that needs to be answered is: How universally valid is the model required to be? More specifically, if we are interested only in one particular process, then perhaps many of its dimensions and other characteristics can be regarded as constants. At the other extreme, if the model is to be generic, representing a very wide class of processes, many more variables might have to be taken into account.

6.3 Modelling a system that exists, based on data obtained by experimentation

A system that exists may be able to produce data from which a model can be constructed. The ideal situation is one where:

(a) The system is available for experimentation with no limits on the amount of data that can be acquired.
(b) The system receives no other signals than those deliberately injected by the experimenter.
(c) The system is, to a reasonable approximation, linear and time invariant.
(d) The system completes its response to a stimulus within a reasonable time scale.
(e) The system has no ‘factors causing special difficulty’.
(f) It is not intended to use the model outside the region of operation spanned by the experiments.

(g) The physical meaning of the model is not of interest.
(h) The only system that is of interest is a unique one, on which the experiments are to be made.

This is a formidable list. It shows why modelling based on experimentation is so difficult. Discussing the points in turn:

(a) Real (for instance, industrial) systems are almost never available for experimentation. This is why pilot plants and laboratory-scale systems are commonly used – unfortunately they are often quite different from large systems in their behaviour with such differences themselves being very difficult to quantify. For this reason, simulations of systems are often used in preference to pilot plants, but of course simulations need system models. However, real systems may usually be observed under normal operating conditions and models may be developed based on the resulting data.
(b) Real systems will usually be subject to operational inputs and unmeasurable disturbances, in addition to any signals originated by the experimenter. The experimenter's signals will always need to observe amplitude constraints and there always arises the question: Is the signal-to-noise ratio of recorded data sufficient to allow modelling to proceed to a level of sufficient accuracy?
(c) Real systems exhibit every sort of undesirable behaviour. Lack of repeatability, hysteresis and asymmetry are the norm and application-specific problems (ASPs) often dominate the project. Additionally, linearity fails for all systems in that increasing the amplitude of applied stimuli will fail eventually to provoke proportional responses. Linearity will often also fail at the other end of the amplitude range, in that, for signals of a sufficiently small amplitude, no output response may be obtained. All of these factors need to be considered when choosing the signals to be injected during an experiment that is specifically designed to produce data for modelling. (Such an experiment will be called an identification experiment.)
(d) It will clearly be convenient if a complete identification experiment can be concluded within a few hours. This will not be possible if the system is very slow to respond to stimuli. The problem will be compounded if an identification method that requires long successions of test signals is used.
(e) Problems in this category are often the most severe from a practical point of view. They include the following:
    (i) Systems that cannot operate except under closed loop control. This situation complicates the identification procedure because some of the system input signals are dependent on the system output signals.
    (ii) Systems where the only practically accessible signals are multiplexed sequential digital signals, often existing as part of a closed loop control system as in (i).
    (iii) Systems where a product forms an essential part of the system, such that experimentation without the product is meaningless and on a small scale is

impracticable. Many industrial processes operate for very long runs and the most important control problems are often intimately linked with the production aspect. For instance, keeping thousands of loaves or steel bars within specification for hour after hour is not something that can easily be emulated on a pilot-scale plant.

(iv) Systems where there are significant trends, i.e. when, in some sense, the mean level of operation changes markedly with time.

(f) Identification may form part of a project that is intended eventually to move the system into a new operating regime. Clearly, a model based on data obtained in one operating region may have little or no validity in a different operating region.

(g) The coefficients in an experimentally based model will owe more to the mechanics of curve fitting than to any physical aspects of the system. This aspect may limit the usefulness of the model since, for instance, it is not possible to estimate from the model the effect of a change in system configuration.

(h) Development projects will often aim to design solutions for a class of systems (rather than for one particular given system). In such instances, it is important not to base global designs on models of only local validity.

6.4 Construction of models from theoretical considerations

A system can most easily be modelled when every aspect obeys established physical laws and where, additionally, all the required numerical coefficients are exactly known. Most usually, real systems have to be heavily idealised before textbook theories can be applied. Such idealisation naturally means that model and system differ appreciably.

Turning to numerical coefficients, these can be classified roughly into three groups:

(i) Universal constants where values are exactly known.

(ii) Coefficients whose role in the theoretical framework is well understood but whose numerical values may vary over a wide range depending on system configuration and prevailing conditions.

(iii) Coefficients on whose numerical values the appropriate accepted theories have little or nothing to say.

6.5 Methods/approaches/techniques for parameter estimation

The methodology for mathematical modelling is as follows. Relevant theories are consulted to yield a tentative set of equations, in which some of the coefficients are unassigned. Data are recorded from particular systems and the coefficients in the equations are adjusted until the set of equations (the model) performs as closely as

possible like the real-world system – as judged by comparison between recorded system data and model-generated data. The comparison is made unambiguous by the definition of a scalar-valued criterion that is to be minimised by choice of model coefficients. Automatic search for the best model coefficients is assisted by parameter estimation algorithms, often called informally, but accurately, hill-climbing methods.

These methods search for the minimum in the multi-dimensional and often ill-conditioned parameter space (ill-conditioned in the sense that the axes are in practice far from orthogonal and the function that is to be minimised often has narrow ridges on which an algorithm without ridge-following abilities may terminate its progress before reaching the minimum).

Figure 6.2 shows the scheme by which observations and model outputs are compared and the difference between them minimised by hill-climbing. Figure 6.3 illustrates the iterative search in parameter space performed by the hill-climbing algorithms.

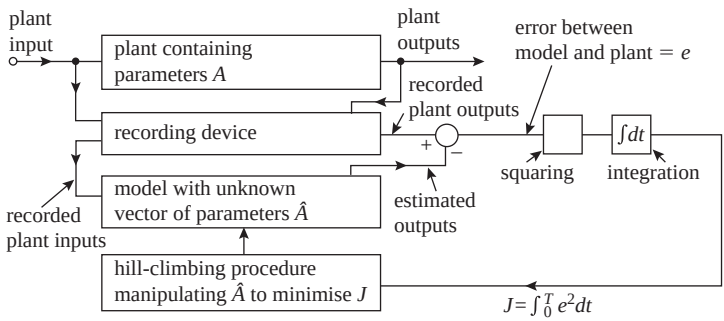


Figure 6.2 The principle of hill-climbing for the estimation of unknown model parameters

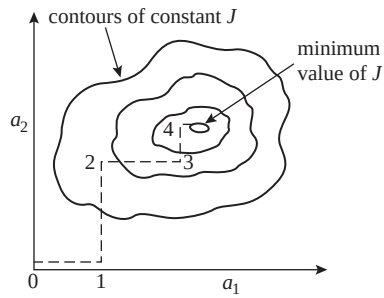


Figure 6.3 Visualisation of an iterative search in parameter space

Rarely, if ever, does the first attempt at modelling succeed in the sense that it produces an accurate usable model. Almost always alternative model structures have to be tried, hill-climbing repeated and the fit between model and reality re-examined, until eventually a sufficiently good model performance is obtained. During the modelling procedure, the misfits between model outputs and measured observations (often referred to as 'residuals') can be plotted to assist in decisions on model changes that might with advantage be made to further improve the fit. In principle, the residuals should contain no deterministic element and should have zero mean – if not, the implication is that there are still unmodelled deterministic features that should be incorporated into the next version of the model.

6.6 Why modelling is difficult: an important discussion

Let Σ be a class of system for which a model M is to be constructed. M is to have a theoretically based structure with experimentally determined numerical coefficients.

It is required that M should represent a large number of actual system examples $S_1, S_2, \ldots, S_n$. To allow the experimental determination of numerical coefficients, sets of operating data are obtained from the ith system S_i.

Each of the different data sets from system S_i can be denoted $D_{ij}, j = 1, \ldots$ and of course, different data sets D_{ij}, D_{ik} may represent nominally identical operating conditions of the system S_i, or they may happen to be different, or they may have been planned to be widely different especially to assist modelling. With the aid of relevant theory, we select particular model structures $M_\alpha, M_\beta, \ldots$ (such selection will always involve a compromise between oversimplification and over-elaboration). Armed with one model structure M_0 and one data set D_{11}, we can use parameter estimation techniques to produce a best fit.

The key question is to what extent is the model structure M_0, with parameters determined from data set D_{11}, meaningful to represent the whole class Σ? It is clear that many data sets from different representative systems would need to be analysed before any claim to universality of models could be made.

The extreme difficulty that this problem represents can soon be appreciated if one thinks of particular examples. Consider, for instance, the modelling of the manufacturing of electronic devices or the modelling of biological growth processes (as required in the manufacture of penicillin). The choice of approach somewhere between theoretically based universality and a practically based one-off solution will depend on the intended use for the model.

A compromise solution to satisfy many short- to medium-term requirements is to find a general tried and tested piece of software that is intended to represent (say) a class of production processes, and then customise it by structural changes and parameter estimation on typical data to represent a particular situation (Figure 6.4).

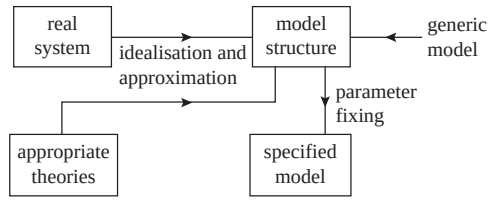


Figure 6.4 The modelling procedure: the route from real system to specified model of that system

6.7 Fixing of parameters

Clearly (see Section 6.4, list (i), (ii), (iii)), some coefficients are universal constants and can be fixed for all time; others are specified by the theory to lie in a known band; for yet others there is no a priori indication of numerical value.

6.8 Parameter estimation

Parameter estimation is the activity of fixing numerical values in the generic model of the system to particularise it to a specific case. From what has already been said, it is obvious that coefficients on which there is no theoretical guidance will need to be specified either by 'case law' (i.e. experience from elsewhere) or by observation/experimentation.

6.9 Regression analysis

(This section is based on Davidson (1965) and, in particular, the graphs in Figure 6.6 are directly quoted from that reference. However, for a very comprehensive, more recent, reference that adequately supports Sections 6.9 and 6.10, see Mendenhall (2008).)

Suppose we assume a mathematical model relating a dependent variable y to a set of independent variables x_1, x_2, ..., x_k:

$$y = a_1x_1 + a_2x_2 + \ldots + a_kx_k \tag{6.1}$$

The a_i are parameters whose values are to be determined from sets of repeated measurements that can be tabulated in the form:

$$\begin{array}{ccccc} y_1 & x_{11} & x_{12} & \ldots & x_{1k} \\ y_2 & x_{21} & x_{22} & \ldots & x_{2k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_n & x_{n1} & x_{n2} & \ldots & x_{nk} \end{array} \tag{6.2}$$

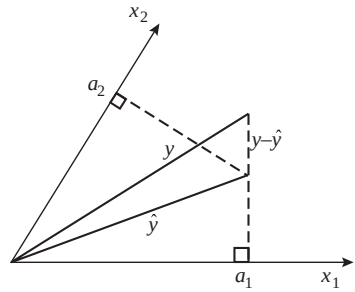


Figure 6.5 The method of least squares considered as the projection of the observed vector onto a k-dimensional hyperplane

or in vector–matrix notation,

$$[y|X] \tag{6.3}$$

where X is an $n \times k$ matrix. It is usually assumed that:

(i) The measurements X have no error.
(ii) Each measurement y has a random normally distributed error, with mean μ and variance σ^2; the variance is the same for all observations, and the errors of the y are statistically independent. There are two approaches to choosing the parameters $a_1, a_2, \ldots$: Gauss's criterion of least squares and Fisher's criterion of maximum likelihood. Under the assumptions listed above, these two approaches lead to the same results.

Minimising the sum of squares between calculated and observed values for y involves solving the set of simultaneous linear equations

$$X^T X a = X_y^T \tag{6.4}$$

leading to

$$a = (X^T X)^{-1} X_y^T = C X_y^T \tag{6.5}$$

As is shown by Davidson (1965), the method of least squares may be viewed geometrically as the projection of the observed vector $y \in \mathbb{R}^n$ onto the k-dimensional observation hyperplane whose basis vectors are the columns of Xa. The projection of y onto the observation space is $\hat{y} \in \mathbb{R}^k$. Figure 6.5 illustrates the concept.

6.10 Analysis of residuals

In a perfect model, the residuals $y - \hat{y}$ display only a random error pattern. Plots of residuals are most valuable in highlighting systematic unmodelled elements.

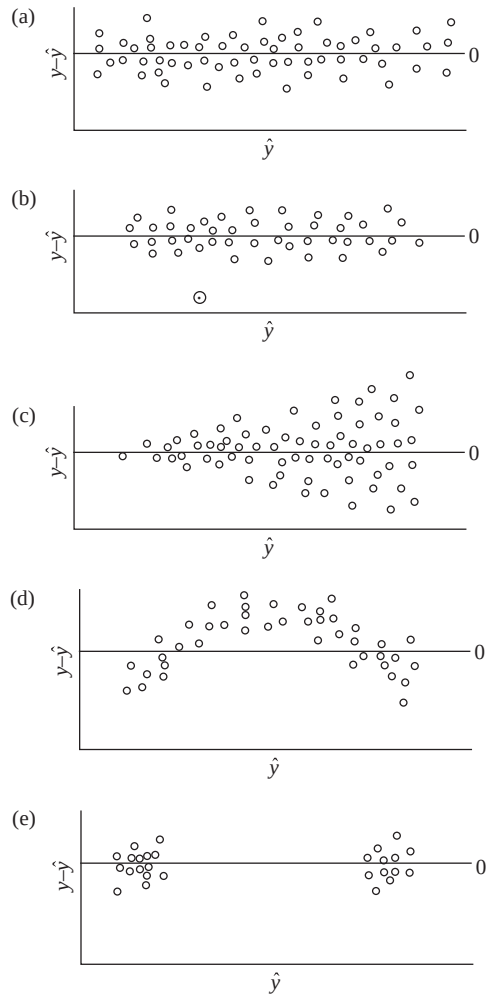


Figure 6.6 Possible plots of residuals (after Davidson (1965))

Figure 6.6 illustrates some of the types of plot that can assist the process of model development and refinement (from Davidson (1965)). Figure 6.6(a) shows the desirable pattern. Figures 6.6(b)–(e) illustrate various types of undesirable bias.

6A Doubt and certainty

An interesting fundamental question that arises in mathematical modelling is to what extent it is ever possible to claim that a particular model structure

is correct. The following extract from Cormack and Mantel (1990) is relevant:

'A theorem in mathematics starts and ends in the mind. Given the initial premises only logic is needed to reach the final answer. But problems arise when the argument starts, not from axioms, but from sense data of the real world. More than one theory will account for the observations and logic may not, by itself, settle the question. In such a case, a well-designed experiment may show which of two contradictory ideas is to be preferred.

A scientific theory is accepted not because it is "true" whatever that may mean, but because it works and is useful. Some helpful rules have emerged. The prime test of a theory is that it should predict correctly. Second, it must be consistent with the rest of science. It must have, as Einstein (French, 1979) put it, both "internal and external coherence". A crucial experiment never verifies the "correct" idea in any absolute sense; and also according to Einstein (French, 1979) "As far as the propositions of mathematics refer to reality they are not certain; as far as they are certain they do not refer to reality".'

6B Anticipatory systems

Anticipatory systems – defined as systems that contain internal predictive models of themselves and/or of their environment, and that utilise the predictions of their models to control their present behaviour – are specially complex from the modeller's point of view (Rosen, 1985).

Systems of this type have a variety of properties that are unique to them, just as 'closed loop' systems have properties that make them different from 'open loop' systems. It is most important to understand these properties, for many reasons. Rosen (1985) argues that much, if not most, biological behaviour is model based in this sense.

This is true at every level, from the molecular to the cellular to the physiological to the behavioural.

Rosen argues:

'An anticipatory system is one in which present change of state depends upon future circumstances, rather than merely on the present or past. As such, anticipation has routinely been excluded from any kind of systematic study, on the grounds that it violates the causal foundation on which all of theoretical science must rest, and on the grounds that it introduces a telic element that is scientifically unacceptable. Nevertheless, biology is replete with situations in which organisms can generate and maintain internal predictive models of themselves and their environments, and utilise the predictions of these models about the future for purpose of control in the present. Many of the unique properties of organisms can really be understood only if these

internal models are taken into account. Thus, the concept of a system with an internal predictive model seems to offer a way to study anticipatory systems in a scientifically rigorous way.

This approach raises new questions of a basic epistemological character. Indeed, we shall see that the utilisation of predictive models for purposes of present control confronts us with problems relating to causality.

The gadgeteers and data collectors, masquerading as scientists, have threatened to become the supreme chieftains of the scholarly world.

As the Renaissance could accuse the Middle Ages of being rich in principles and poor in facts, we are now entitled to enquire whether we are not rich in facts and poor in principles.

Rational thought is the only basis of education and research. Facts are the core of an anti-intellectual curriculum.

One of the best-studied biological homeostats is one involved in maintaining an optimal constancy of light falling on the retina of the vertebrate eye, the so-called "pupillary servomechanism". Roughly speaking, in conditions in which there is a great deal of ambient light, the pupil contracts, and admits a smaller amount of light to the eye. Conversely, when the ambient light is dim, the pupil opens to admit more light. It has been established that the control system involved here is a true feedback system, whose output is represented by the actual amount of light falling on the retina.

Thus, the sensor for the controller is at the retina, and the system reacts to how much light has already been admitted to the eye. The time constant for this servomechanism is not outstandingly small, but the system clearly functions well for almost all conditions that the organism encounters.

Now let us consider the analogous problem of controlling the amount of light entering the lens of a camera to ensure optimal film exposure. Here again, the control element is a diaphragm, which must be opened when the ambient light is dim, and closed when the ambient light is bright. However, in this case, we cannot in principle use a reactive mechanism at all, no matter how small its time constant. For clearly, if the input to the controller is the light falling on the film, in analogy to the situation in the eye, then the film is already under- or over-exposed before any control can be instituted. In this case, the only effective way to control the diaphragm is through an anticipatory mode, and that is what in fact is done. Specifically, a light meter is then referred to a predictive model, which relates ambient light to the diaphragm opening necessary to admit the optimal amount of light to the camera. The diaphragm is then preset according to the prediction of the model. In this simple example we see all the contrasting features of feedforward and feedback; of anticipatory as against reactive modes of control. [This note has been added by the author JRL: 'since those words were written, intelligent flashguns have become available that, working closely with a coupled camera, do work in feedback mode as follows. The lens is opened, the flash begins, and light is reflected from the subject back into the lens to make the exposure and to be monitored and integrated by a through-the-lens light

meter. When calculation shows that exposure is complete, the flash is terminated. The high velocity of light makes this remarkable feedback loop possible. To complete this discussion, I note that the latest Nikon Speedlight can control its length of flash over the range from zero to a maximum of around 1/1000 second to make this feedback operation possible.']

If it were necessary to try to characterise in a few words the difference between living organisms and inorganic systems, such a characterisation would not involve the presence of DNA, or any other purely structural attributes; but rather that organisms constitute the class of systems which can behave in an anticipatory fashion. That is to say, organisms comprise those systems which can make predictive models (of themselves, and of their environments) and use these models to direct their present actions.

At the most fundamental level, anticipatory systems appear to violate those principles of causality which have dominated science for thousands of years. It is for this reason that the study of anticipatory systems per se has been excluded routinely from science, and that therefore we have had to content ourselves with simulations of their behaviour, constructed in purely reactive terms.'

6C Chaos

Smale and Williams (1976) showed that non-linear dynamic systems of order 3 or more may exhibit chaotic behaviour, first identified by Li and Yorke (1975). Chaotic behaviour is characterised by the following:

(i) Any individual solution has a completely well-defined deterministic trajectory.
(ii) Very small perturbations, for instance to the initial conditions, can give rise to very large differences between later trajectories.
(iii) Solutions of equations exhibiting chaotic behaviour may be difficult or impossible to distinguish from solutions generated by a purely stochastic process.

The difference equation

$$x(k+1) = rx(k)(1 - x(k)) = fx(k)\ (say) \tag{6.6}$$

can also exhibit chaotic behaviour as the parameter r is varied. (This is because the delay term implicit in a difference equation represents infinite dimensionality, as judged, for instance, by the order of s plane poles.)

There are two equilibrium points

at $x = 0$ and $x = 1 - \frac{1}{r}$ Behaviour of (6.1):

Equation (6.1), which arises in population dynamics, will be studied for the restricted set of values $0 < x < 1$. The behaviour of (6.1) may be understood graphically, using repeatedly a curve relating $x(k+1)$ to $x(k)$ as in Figure 6.7.

For use, $x(k+1)$ is derived from $x(k)$, then

$x(k+2)$ is derived from $x(k+1)$, etc. (6.7)

The process can be simplified using a 45° line to transfer each ordinate value back to the abscissa to start the next iteration as shown (Figure 6.8). (Local) Stability depends on the slope off near to the equilibrium point. This slope f' must satisfy

$$|f'| < 1 \tag{6.8}$$

$$\frac{df}{dx} = r(1-2x) \tag{6.9}$$

and at the non-trivial equilibrium point $1 - 1/r$

$$\frac{df}{dx} = r\left(1 - \left(2 - \frac{2}{r}\right)\right) = 2 - r \tag{6.10}$$

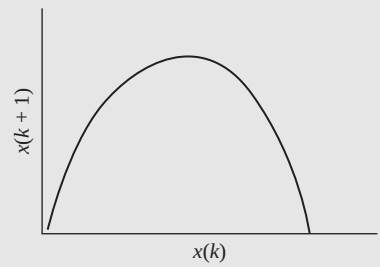


Figure 6.7 The curve relating x(k + 1) or x(k) (relevant to (6.1))

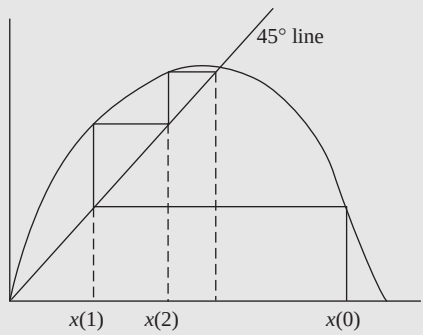


Figure 6.8 Graphical illustration of the iterations in the solution of (6.1)

Thus, the non-trivial equilibrium point is (locally) stable if

$$1 < r < 3 \tag{6.11}$$

For $r > 3$, the solution is initially one that, in the steady state, oscillates between two fixed points. When r is increased further the system oscillates between 4, 8, 16, etc. fixed points. These stable oscillations with periods 2^n, $n \to \infty$, continue only up to a critical value r_c of r. For (6.1), $r_c = 3.57$. For $r > r_c$, very long cycles appear and different types of periodic behaviour are passed through. Between this type of behaviour occurs another type of behaviour where different initial points produce different totally non-periodic behaviour. It is this non-periodic behaviour that is called chaotic behaviour.

Segel (1980), Chapter 4, states that, for any particular value of parameter r, the set of initial conditions that gives rise to chaotic behaviour has measure zero. According to this, chaotic behaviour is atypical and should not therefore be considered as the obvious source of erratic behaviour in observed data. The most interesting aspect of all the foregoing is probably that very simple equations can give rise to highly complex solutions.

6D Mathematical modelling: some philosophical comments

It can be rewarding to glance sometimes beneath the mechanistic surface activity of mathematical modelling to query the hidden foundations. Here we content ourselves with the following brief discussions:

(1) *On causality and time-ordering*

Causality causes an awkward asymmetry in mathematics. Time hardly appears in pure mathematics and, where it does, anti-causality would be just as valid, feasible and usable.

Difficulties are most likely to be encountered when synthesising an optimal controller or an algorithm for reconstructing a continuous time signal from given discrete samples.

As an illustration, let $y(t)$ be a continuous signal defined for all real vales of t and let $y^*(kT)$ be the properly sampled version of the same signal. If now, given some specific t, it is required to recover $y(t)$ from the sequence of samples, the recovery algorithm will be found to have the form

$$y(t) = f \sum_{k=-\infty}^{\infty} y^*(kT) \tag{6.12}$$

for which, when used as a real-time algorithm, only current and past values of $y^*(kT)$ can be available.

It would be desirable, but is not always possible, to insert a priori conditions into derivations to ensure that the solutions will be causal and therefore implementable. In transfer function manipulation, causality is ensured by simply outlawing as anti-causal, any transfer function whose numerator has a higher order than the denominator.

(2) *Time-ordering*

Aulin (1989): 'But sometimes the time-ordering between cause and effect is left unspecified, and only implied. Examples of this kind of causal law are Ohm's law, Coulomb's law, Biot–Savart's law and the laws (Boyle, Gay-Lussac, etc.) that characterise the thermodynamic equilibrium. Examples of time-specified causal laws are, of course, plenty. Among them are the law of the freely falling body and other laws of mechanics, as well as the laws of electrodynamics. Common to all laws of physics mentioned above is that they are "phenomenological laws", i.e. more or less conceived of as direct inductive generalisations from experimental results (or, if they are not, they can still be considered as such generalisations).'

(3) *On the surprising simplicity of the mathematics that suffices to model very complex physical systems*

Dietrich (1994): 'This is the old question about the unreasonable effectiveness of mathematics in the natural sciences or as Davies put it "why the universe is algorithmically compressible" (i.e. why the obviously complex structure of our world can be described in so many cases by means of relatively simple mathematical formulae). This is closely linked to why induction and therefore science at all, succeeds. It is difficult to avoid asking whether mathematics, as the outcome of human thinking, has its own specificity which, for whatever reason, fits to the specificity of what man would see or experience.

As long as this question is not comprehensively answered, science may explain much but not its own success.'

See also Eugene Wigner (1960) on 'The Unreasonable Effectiveness of Mathematics in the Natural Sciences.'

It seems that the Creator had only a few simple mathematical equations with which to underpin the immensely complex phenomena that the Universe contains. There are hundreds of illustrative examples of which the best known is possibly the law of gravity that Newton postulated based around very sparse and not very accurate observations of falling bodies and of the motion of the Moon's path through the sky. Newton's laws fitted the few available observations of the time to within about 4%. As observations have become much more accurate and more numerous, it has been found that Newton's gravitational law is accurate to better than 1/10,000th of 1%.

A somewhat different illustration but equally impressive is the case of Maxwell's equations (1862) (see Fleisch (2008)) describing the magnetic field.

Largely for reasons of symmetry, Maxwell enhanced the equations with an expression that predicted the existence of electromagnetic waves that were unknown at the time. When Maxwell published his findings that electromagnetic waves may exist and propagate through free space, there was no way to verify that finding. However, there was available an approximate value at that time for the velocity of light and this was so close to the value calculated by Maxwell for his electromagnetic phenomenon that he wrote: 'It is scarcely possible to avoid the inference that light consisted of transverse undulations of the same medium that is the cause of electrical and magnetic phenomena'. In 1887, Hertz experimentally verified the existence of the electromagnetic waves predicted by Maxwell. (See Baigrie (2007) for more details on this topic.)

(4) *On determinism and predictability*

Strong determinism: the predictability, with certainty, of single future events in the given dynamical system.

Probabilistic determinism: the predictability of the probability distributions of future events.

Weak determinism: the predictability of the possibility distributions of future events.

Interdeterminism: the unpredictability of all future events in the dynamic system concerned.

Thus, the concept of causality cannot be identified simply with 'determinism', but allows three different degrees of determinism and, in addition to them, a case of complete indeterminism.

(5) *On reversibility and irreversibility*

What is the general quantitative measure of irreversibility? Nature does not permit those processes for which she has less predilection than she has for the initial states. The measure of nature's predilection was defined by Clausius as entropy.

Consider the differential equations

$$\frac{d^2y(t)}{dt^2} + y(t) = 0 \tag{6.13}$$

$$\frac{d^2y(t)}{dt^2} + \frac{dy}{dt} + y(t) = 0 \tag{6.14}$$

The first equation can be seen to represent a reversible process that will have a similar solution for both t and $-t$. The second equation is stable for positive time but unstable for negative-going time. The lesson from this simple example is generalisable so that differential equations with only even-order terms can be expected to represent reversible processes.

(6) *On modelling social and political phenomena*

At a deep enough level, both the arts and the sciences are seeking for meaning. At that level, do the arts and the sciences begin to merge?

Quoting Truesdell (1984): 'Nothing is easier to apply to socio-political phantasmagoria than failed mathematics substantiated by experiments programmed to confirm it.' And

'Rarely if ever does a scientist today read Newton and Euler as professors of literature read Shakespeare and Hemingway, seeking to translate into today's pidgin for their students the eternal verities archaically expressed by those ancient masters, or gathering material to use in papers for reverential journals read only by scholiasts (sic) of literature, who themselves read only to gather material to help them write more papers of the same kind.'

6E A still relevant illustration of the difficulty of mathematical modelling: the long march towards developing a quantitative understanding of the humble water wheel, 1590–1841

In this example, excessive reliance on a scientific theory that didn't quite apply significantly hindered quantitative understanding of the key phenomena involved.

In Britain in the eleventh century there were, according to the Domesday book, 5624 water mills; by the eighteenth century the number had increased to as many as 20,000. Water wheels were of great economic importance in most of Europe over many centuries since they provided the bulk of the power for many basic installations (mining, metal forming, milling) and they were also used to pump water, with notable examples being their use on the Seine at Marly where 14 water wheels lifted water 502 ft to supply fountains, gardens and palaces, including Versailles.

By the eighteenth century, there was considerable overcrowding of water wheels on many European waterways and in many locations, no more wheels could be fitted in. Thus, there was a strong incentive to design water wheels of maximum efficiency.

The Problem

Few mechanisms seem easier to understand 'by inspection' than a basic water wheel. There are two types – 'undershot' (when the wheel dips into a stream or mill race) and 'overshot', where a duct feeds the water over the top of the wheel, which then turns by the force of gravity (Figure 6.9).

Although it is obvious in the extreme how water wheels work and although nothing is hidden from our view and all the laws of gravity, force,

etc. are, and were, well known, the development of quantitative understanding contains salutary lessons.

Which is the more efficient, the overshot or the undershot wheel?

A theorem of Torricelli of 1643 states that water spouting from an orifice at depth H in a tank and water in free-fall for a vertical height H both have identical velocities (Figure 6.10). (Evangelista Torricelli (1608–1647) was an Italian mathematician and physicist who worked closely with Galileo and who gave his name to the Torricellian vacuum at the top of a mercury in glass barometer.)

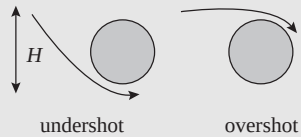


Figure 6.9 Modelling a water wheel. There were 15,000 water wheels in Britain in the 1700s. Among those who studied it then were Huygens, Maclaurin, Euler, Navier, Coriolis, Lagrange and D'Alembert. There were huge discrepancies between theory and observation. The typically British 'method of coefficients' overcame this but made it difficult to know what, in the design, was significant. Accurate models only became available when systematic (expensive) experimentation was undertaken by the Franklin Institute, around 1830, by which time, steam was replacing water power

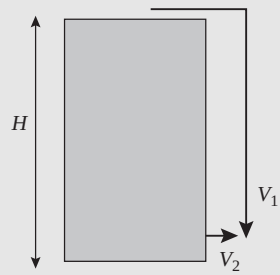


Figure 6.10 Modelling a water wheel. The theory that misled: the velocities V_1 and V_2 are equal; $V_1 = V_2 = \sqrt{2gH}$ (Torricelli, 1643)

This theorem is correct but it misled a series of scientists into wrongly assuming that impulse and weight were equally effective as motive powers and therefore that both types of water wheels (undershot, overshot) must necessarily have the same efficiency. Some of Europe's most distinguished scientists made armchair pronouncements purporting to be the defining relations for both types of wheels. Most of these pronouncements turned out to be well wide of the mark, as some sample quotations show (Table 6.1).

Table 6.1 Analyses of the vertical water wheel, c. 1700–c. 1800

Investigator	Date	Maximum possible efficiency of wheel	
		Undershot	Overshot
Parent	1704	15	15
Euler	1754	15–30	100
Borda*	1767	50	100
Bossut	1770	15	
Waring	1792	25	
Evans	1795	58	67
Buchanan	1801	59	

*Borda's analysis proved eventually to be substantially correct but this was not verified and accepted until 70 or 80 years had passed.

As Table 6.1 indicates, there were many rival theories producing quite different conclusions.

All real progress towards understanding was made on the basis of experimentation, and in particular, in England, the land of pragmatists, an approach called 'the method of coefficients' had begun to be applied. The method was to multiply terms in theoretical equations by numerical coefficients to make theory agree with practice. Thus, two opposing views prevailed:

- That of the British camp typified by Milner (1778) who said: '(Continental) writers who published water wheel analyses really had no intention of making any improvements in practice. They were simply illustrating the use of algebra or the calculus. Too many arbitrary assumptions were made for them ever to correspond with reality.'
- And that of the continental theorists who complained that inexactness was inherent in coefficient equations, since resistance, friction and all other losses were taken as a block and expressed by a constant coefficient. Every loss, they argued, depended on different circumstances, and could not be expressed by a single constant relationship.

Since all losses were included in one figure, it was impossible to study the influence of each on the wheel's performance.

So theoreticians continued to derive ever more complicated equations, pushing the mathematical analysis of the vertical water wheel to new limits, while practising engineers used the so-called method of coefficients in which experimentally derived coefficients were inserted into basic theoretical equations to bring them into close agreement with practice.

By 1835, the steam engine had arrived on the scene and had taken over more than 50% of industrial applications. As an anticlimax, by around 1850, extensive experiments had finally allowed the working-out of a fairly complete theory of water wheel operation and an understanding of the effects of various design features of performance.

In summary:

- Quantitative understanding of real processes is very difficult.
- Theory rarely (i.e. never) applies easily in an application context.
- Experimentation is difficult to plan or interpret without a theory.

(Milner I. 'Reflections on the communication of motion by impact or gravity'. *Philosophical Transactions of the Royal Society of London.* 1778;**68**:344–79)

6F A postscript on the effectiveness of thought experiments

(1) *A thought experiment about the water wheel*

One of the most impressive steps forward in the development of understanding of water wheel operation, see interlude 6E above, was made by a thought experiment by de Parcieux (1754). He imagined a very slowly rotating frictionless water wheel gravity driven by dripping water. He was able to argue convincingly that no inevitable losses would occur in such a system and that the efficiency for an overshot wheel could therefore approach 100%, which turned out to be the case (the work of de Parcieux, which was carried out in the 1750s has been i described as 'a triumph of experiment over theory' and it has been quoted here because it contains some salutary lessons to all would-be modellers. De Parcieux' work is described in some depth in chapter 4 of Reynolds (1983)).

(2) *Another success for thought experimentation; conjecturing about the International Date Line*

On Thursday, 10 July 1522, the Portuguese explorer Ferdinand Magellan completed one of the earliest circumnavigations of the world, and on his arrival back in the Cape Verde Islands, he and his crew were amazed that they had 'lost a day', since according to their carefully kept log, the day was Wednesday, 9 July.

Among many people who conjectured over this anomaly was Charles L. Dodgson (Lewis Carroll) who much later (1860) argued along these lines: 'Imagine that all the Earth were land and that a person could run right round the globe for 24 hours with the sun always overhead. That person would never see the sun rise or set. However, at the end of the trip the person would be at the same point he or she started from, but, 24 hours having elapsed, the day must have changed. So the question arises: At what point in the journey did the day change?

Dodgson's simple argument or 'thought experiment' makes very clear the need for some line where the date would change. (The International Date Line came into being only in 1884.)

6G Experimentation on plants to assist in model development: the tests that you need may not be in the textbook!

In order to allow simulation of different scenarios for a not-yet-built automation scheme, it was necessary to know the load characteristics (inertia and friction as a function of angular velocity) of an existing composite gear train that was to be used in the system. The system, Figure 6.11, consists of a 30 kW motor driving a massive load through a gear train of about 1400:1 reduction.

This must be a common problem but the only reference found (Libby, 1960) was unhelpful. Acknowledged experts on mechanical drives who were asked to help, sketched expected curves that later were shown to be qualitatively well wide of the mark.

The following simple test, inspired by an undergraduate laboratory experiment, provided all the information needed. (Credit for the basic idea must be given to the unknown author of the Cambridge University Electric Machines lab. sheet of the day) The DC electric drive motor is switched on to the supply at voltage v and its steady state current i and steady angular velocity ω_{max} are recorded.

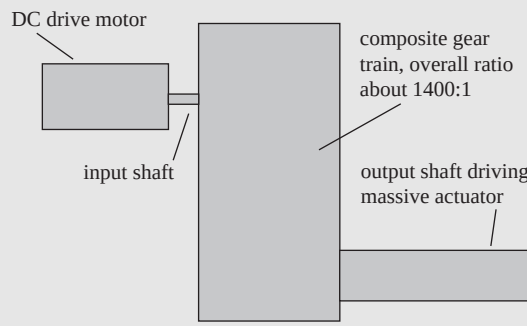


Figure 6.11 The motor and load whose overall inertia and torque as a function of angular velocity were determined experimentally

It is then argued that in the steady state,

$$\text{(electrical power to the motor} - \text{losses in the motor)} = \text{mechanical power delivered to the input shaft} \tag{6.15}$$

Or

$$vi - \text{motor losses} = \omega_{max} T(\omega_{max}) \tag{6.16}$$

where $T(\omega)$ denotes the resisting torque of the load at angular velocity ω.

Leaving out motor losses for purposes of this explanation (since the principle is unaffected) allows calculation of $T(\omega_{max})$ as

$$T(\omega_{max}) = \frac{vi}{\omega_{max}} \tag{6.17}$$

Next we switch off the motor and record the decay of ω against time (Figure 6.12).

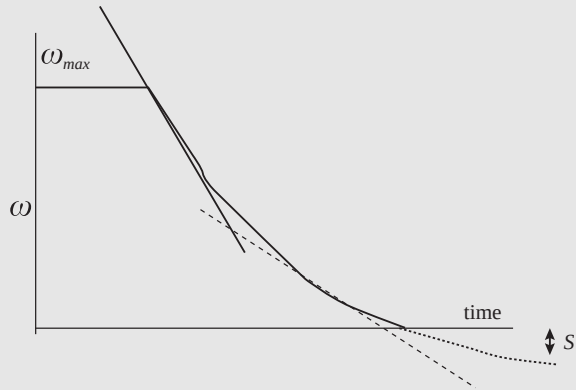


Figure 6.12 Illustrating how dω/dt as a function of ω is estimated by tangents to the experimental curve. Notice also how the estimate S (not discussed in the text) is a useful measure of static friction in the drive

The argument now is that at switch-off, the load torque $T(\omega_{max})$ is the only agent that slows the shaft, whereas the effective inertia, call this J, of the whole load as seen at the input shaft is the agent that continues to drive the load in the absence of power being applied.

The relevant equation is

$$j\frac{d\omega}{dt} + \omega T(\omega_{max}) = 0 \tag{6.18}$$

The inertia J, assumed invariant for all ω, can be found from

$$J = \frac{-\omega_{max} T(\omega_{max})}{(d\omega/dt)_{\omega=\omega_{max}}} \tag{6.19}$$

and by drawing the solid tangent shown in Figure 6.12, the inertia J can be derived. (In the case described here, a laborious day's work by the author, working on engineering drawings and referring approximate inertias all the way through the composite train, produced a confirmatory figure only 8% away from the experimental figure.)

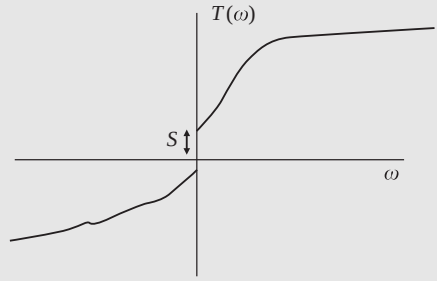


Figure 6.13 The final torque versus ω curve has this form. Here S denotes static friction (see Figure 6.12)

Next a sequence of tangents (shown dotted in Figure 6.12) was drawn at frequent points along the ω decay curve and at each chosen ω. The load torque at each of the chosen ω was then calculated from

$$T(\omega) = \frac{-J(d\omega/dt)}{\omega_{\omega chosen}} \tag{6.20}$$

allowing the curve of $T(\omega)$ to be plotted against ω (Figure 6.13). In use, it was stored as a look-up table interpolated by a subroutine at every step in an overall process dynamic simulation.

6H Dimensional analysis

Both sides of an equation involving physical quantities must necessarily have the same dimensions when expressed in terms of the agreed fundamental units of mass, length, time, electric current and temperature. Dimensional analysis (see Bridgman (2008)) is a useful methodology that relies on this fact in a way that can help model building as illustrated below.

Imagine meeting a pendulum for the first time* and wondering what formula might provide a way of calculating the period T, using an equation of the form $T =$ (right hand side: structure to be determined).

*Galileo is said to have been one of the first observers to realise the useful time-keeping properties of pendula. During a long service in Pisa cathedral, he noticed, using his pulse as a clock, that the period of a slowly swinging lamp remained constant despite the decaying amplitude of swing. He incidentally confirmed the old adage that a good scientist doesn't always need expensive kit!

Clearly T must have units of time, seconds (say). We ask how those units can be provided in the RHS of the equation. We have only L and M, the length and mass of the pendulum as obvious physical variables, but they can provide only metres and kilograms.

With brilliant hindsight, we suddenly think, what about using g with its units of metres/(seconds)^2. We can't make any dimensional use of M, g will have to be in the denominator to bring seconds to the top line. We shall need a square root to get to seconds from (seconds)^2 and L will have to be used to cancel the unwanted metres of g. This thought experiment therefore leads us to the equation:

$$T = k\sqrt{\frac{L}{g}} = (\text{dimensionally})\sqrt{\frac{ms^2}{m}} \tag{6.21}$$

with units of seconds on both sides.

Of course, a physically informed analysis will provide an understanding of the behaviour of the pendulum but the point to note is how much could be achieved just by considering dimensions, with no consideration of the mechanism or forces and accelerations involved.

Next we take an approximate look at the comparative digestive systems of a mouse and a cow, supposing that a cow might be about 40 times bigger than a mouse in linear terms. That means that its surface area would be about 1600 times greater and its weight about 64,000 times greater. I postulate that the surface area of the cow's intestines should be proportional to its weight but if a cow's interior was just a scaled-up version of that of the mouse, the surface area of the cow's intestines would be 1600 times greater than those of the mouse, whereas to be proportional to weight, they should be 40 times greater than that. In fact, a cow's small intestine is about 20 times longer than the animal, whereas that of the mouse is probably about half the length of the animal; theory appearing to tie up with facts.

Model builders also need to be interested in the question of scale-up, which appears whenever small-scale experiments and pilot plants produce results that must be extrapolated to apply to industrial-scale plants. Most demonstrative of scale-up effects is perhaps that of attempting to use ship models in small tanks. Here, the equation describing the power needed to keep a ship moving in a straight line at some constant speed is a function of the two dimensionless parameters, Froude number and Reynolds number. The first contains g and models the power consumed by wave generation, the second models viscous resistance. If a 1/100th size scale model is to be used of a particular ship, then the model would have to move at 1/10th of the speed of the full-scale ship to keep the same Froude number, but the fluid used in the tank would need to have a Reynold's number 1/1000th of that of water and no such fluid exists. (Air has a viscosity 15 times greater than that of water.)

6I Distributed systems: the cosy idealisation that a system can be characterised by its behaviour at a point in space

Because of exposure to school physics and what in the United Kingdom is called applied mathematics, we are conditioned to accept without question that, for instance, an object, missile or projectile, flying through space, can be truthfully represented by a single point located at the object's centre of mass. This practice, while allowing neat examination questions, leads us into a false sense of simplistic security. For instance, as soon as a projectile is made to spin about its axis of travel (a common practice), we may be unprepared for the escalation of complexity of the problem that this simple addition to the problem causes.

Physically large systems can rarely have their characteristics approximated at a point in space without severe and often unacceptable levels of approximation. It seems to be a very interesting law of nature that increased size brings increased non-uniformity.

For instance, a small sample of the Earth's atmosphere, say a few metres square, will be approximately uniform. However, seen on a scale of hundreds of kilometres, there is extreme non-uniformity in the atmosphere, with discrete cloud forms separated by cloudless atmosphere and there are gusting winds interspersed by calm regions.

Given a system whose spatial behaviour needs to be modelled, there are three possible approaches:

(1) To model the global behaviour by a single set of partial differential equations. Solution is then obtained by numerical methods that, depending on discretisation, approximate one partial differential equation by a set of ordinary differential equations.

(2) To spatially discretise the physical problem into regions within which the behaviour can, with sufficient accuracy, be representable at a point. For each region, an ordinary differential equation is needed. This equation is formulated, identified and solved in the usual way for such equations.

Note, however, that, when the solution from the set of differential equations is patched together to yield the overall system solution, there may be spurious results generated at the (physically non-existent) boundaries that separate the notional regions used in the discretisation. Rosenbrock and Storey (1966) has illustrated spurious results of this sort.

(3) 'Fourier type' modelling in which the distribution is modelled approximately but to any required degree of accuracy by a weighted sum of basis functions f_i. More specifically, if the function to be approximated on the interval $[x_0, x_1]$ is $g(x)$, then scalars α_i are chosen to minimise

$$\int_{x_0}^{x_1} \left(g(x) - \sum_{i=0}^{n} \alpha_i f_{i(k)} \right)^2 dk \qquad (6.22)$$

Preferably the basis functions f_i satisfy

$$\langle f_i f_j \rangle = 0, \quad i \neq j \tag{6.23}$$

i.e. they are orthogonal. This produces the (related) practical advantages: there is no danger of the series becoming ill-conditioned and the values of α_i do not depend on n; i.e. let $\sum_{i=1}^{3} \alpha_i f_i$ be the best third-order fit to some given function $g(x)$ then the best fourth-order fit $\sum_{i=1}^{4} \alpha_i f_i$ to $g(x)$ will have unchanged α_i, values for $i = 1$ to 3.

Alternative approaches to the modelling of distributed systems

The representation of a spatial region as the summation of elemental regions

This approach, familiar to all who have studied mathematical physics, proceeds by defining a small element of dimension δx, δy and δz and then using equations of conservation and continuity, in conjunction with the usual methods of calculus, in which the size of the element is reduced by a limiting process to have dimension dx, dy and dz, to obtain a partial differential equation in the four variables x, j, z, t. The approach produces classical partial differential equations that have been extensively studied and that have known solutions.

Difficulties that may be encountered are as follows:

(i) The region under study may not divide naturally into regularly shaped elements so that approximations or awkward accommodations at the boundaries may have to be made.
(ii) The 'natural' element spatial regions will often, in an industrial application, be variable shapes that may change position.
(iii) Numerical solutions will nearly always involve a return to approximation of the region by a finite number of discrete regions, in each of which an ordinary differential equation governs the local behaviour.
(iv) Fictitious discontinuities – present between the regions defined in (ii) above but not present in the real process – may cause spurious effects, such as travelling waves, to appear as part of the model behaviour.
(v) For a typical industrial process whose detailed mechanisms are very complex, it will be the preferred approach to set up a simple model whose structure is determined from theoretical considerations and whose coefficients are found numerically using parameter estimation techniques on process data. Such a modelling identification procedure is difficult or impossible to carry out on most real processes, using a classical partial differential equation approach.

A 'Fourier type' approach, in which an arbitrary function f on an interval [0, 1] is approximated by a summation of functions f_i

We postulate that

$$f = \sum_{i=0}^{\infty} c_i f_i \tag{6.24}$$

Where f is the function to be approximated on [0, 1], f_i are *basis functions*, each defined on [0, 1] and the c_i are scalar-valued coefficients.

Many questions immediately arise:

(i) Under what conditions on f and f_i will the infinite series be convergent?
(ii) Define $f_n = \sum_{i=1}^{n} c_i f_i$. We ask: Can f_n be used as a reasonable approximation to f? Can we obtain an error estimate for $f - f_n$? Can we, operating with f_n instead of f, still work within a sound theoretical framework?
(iii) What choice of functions f_i will form a basis for the function space?
(iv) What choice of functions f_i will be numerically convenient and widely applicable (we have in mind *orthogonality* (is it necessary?) and behaviour at the end points 0 and 1 (we would like to avoid the enforced condition, typical of Fourier series that, necessarily, $f(0) = f(1)$)).
(v) Is it an advantage if the functions f_i are the eigenfunctions of some operator? If so, can that operator be found in a real situation?
(vi) Do the set of functions $\{f_i, i = 1, ...\}$ form a state in the rigorous sense?
(vii) How may the coefficients c_i be determined from numerically logged process data?
(viii) Can an equation $\dot{x} = Ax + Bu, x \in X, u \in U$, where X is the set of system states, U is the set of input functions and A, B are operators, be set up, identified and used analogously with the usual finite dimensional control equation of the same form?
(ix) To what extent can the theory of operators, compact operators, closed operators, self-adjoint operators and semi-groups be usefully exploited?
(x) Can specific use be made of the projection theorem whereby a function (infinite dimensional) is approximated by its projection onto a finite dimensional subspace?

6J When can the behaviour in a region be well approximated at a point?

An interesting question is: are there fundamental guidelines to help the decision on whether a given situation can be well-approximated by the behaviour at a point? (If such guidelines can be found, they might be

extremely useful in helping to choose the size and shape of regions, when spatial discretisation does turn out to be required.)

One such guideline, attribute to Roots (1969), is as follows:

Let $f_{\max}$ represent the highest frequency of interest to which a spatial region is subject. Let l represent the largest physical distance in the region. Then provided that

$$l \ll 1/f_{\max} \tag{6.25}$$

a point representation (i.e. an ordinary differential equation model) will be justifiable. The argument appears to be that, if the physical size of the region to be modelled is much smaller than the shortest wavelength of externally applied stimuli, then the speed of propagation of effects may be regarded as instantaneous.

The relationship proposed above leaves a number of unanswered questions. For instance, in the heating of a solid object, the thermal conductivity of the material would clearly influence the uniformity of temperature that would be achieved under conditions of externally applied periodic heating stimuli, yet the proposed relation can take no account of this.

Even where a situation can be modelled exactly by unapproximated partial differential equations and the solution is obtained analytically, there is still a possible anomaly in that (for instance) the temperature distribution in a long bar is supposed to evolve as shown in Figure 6.14.

That is, the implication is that the speed of propagation is infinite (see John (1975), pp. 175, 176).

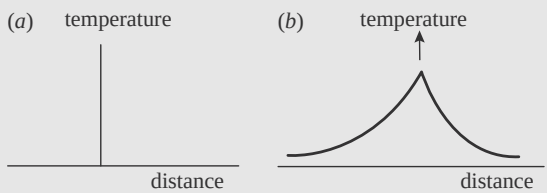


Figure 6.14 (a) A supposed initial temperature distribution in a long bar at $t = 0^-$; (b) the form of the temperature distribution at $t = 0^+$

6K Oscillating and osculating approximation of curves

For Fourier series and for other series of orthogonal functions (Hermite, Laguerre, Legendre, etc.) the approximating series approaches the required function through closer and closer oscillations. In marked contrast, the Taylor

series approaches the required function by osculating at the point around which the expansion is being made. At that point, the approximation and the function approximated have exactly the same derivatives up to and including the nth derivative, for an nth order Taylor series. Figure 6.15 shows successive terms of a Taylor series being fitted to the function sin x. This section follows Sommerfeld (1949), which should be consulted for further details.

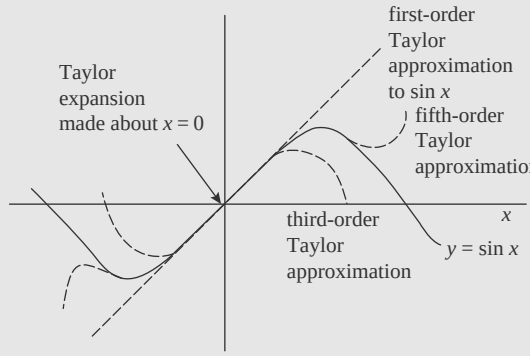


Figure 6.15 The approximation of sin x by three different orders of Taylor series expansions

Chapter 7

Non-linear systems

7.1 What is meant by non-linearity

If John gets 10 hectathrills by taking to a ball a lady aged 24 and of height 5 ft 5 in., how many hectathrills would he obtain by taking to the same ball a lady of height 10 ft 10 in. and aged 48? (With apologies for the failure to use SI units and with acknowledgments to Linderholm (1972)).

In the linear world, the relation between cause and effect is constant and the relation is quite independent of magnitude. For instance, if a force of 1 N, applied to a mass m, causes the mass to accelerate at a rate a, then according to a linear model, a force of 100 N, applied to the same mass, will produce an acceleration of $100a$.

Strictly, a linear function f must satisfy the following two conditions, where it is assumed that the function operates on inputs $u_1(t)$, $u_2(t)$, $u_1(t) + u_2(t)$, $\alpha u(t)$, where α is a scalar multiplier.

(i) $f(u_1(t)) + f(u_2(t)) = f(u_1(t) + u_2(t))$

(ii) $f(\alpha u_1(t)) = \alpha f(u_1(t))$

Any system whose input/output characteristic does not satisfy the above conditions is classified as a non-linear system.

Thus, there is no unifying feature present in non-linear systems except the absence of linearity. Non-linear systems sometimes may not be capable of analytic description; they may be sometimes discontinuous or they may contain well-understood smooth mathematical functions.

The following statements are broadly true for non-linear systems:

(i) Matrix and vector methods, transform methods, block-diagram algebra, frequency response methods, poles and zeros and root loci are all inapplicable.
(ii) Available methods of analysis are concerned almost entirely with providing limited stability information.
(iii) System design/synthesis methods scarcely exist.
(iv) Numerical simulation of non-linear systems may yield results that are misleading or at least difficult to interpret. This is because, in general, the behaviour of a non-linear system is structurally different in different regions of

state space (where state space X is defined for a non-linear system according to the equation

$$\dot{x} = f(x, u) \tag{7.1}$$

$$y = g(x) \tag{7.2}$$

and

$$x \in X \tag{7.3}$$

where the n-dimensional state vector x can be visualised as being made available for control purposes by a non-linear observer with inputs u and y and with output $\hat{x}$, where as usual the superscript indicates an estimated value).

Thus, the same system may be locally stable, unstable, heavily damped or oscillatory, according to the operating region in which it is tested. For a linear system, local and global behaviour are identical within a scaling factor – they are topologically the same. For a non-linear system, it is generally meaningless to speak of global behaviour.

Very loosely, we can organise our thinking about non-linearity in real-world systems with the aid of Figure 7.1. We comment as follows:

(a) very few real-world systems are strictly linear;
(b) a large class of systems can be regarded as approximately linear;
(c) a strongly non-linear class exists, but such systems may often be linearised;
(d) a class whose non-linearity is its most important characteristic exists and needs special consideration.

Linear methods will normally be applied to class (b) without any discussion.

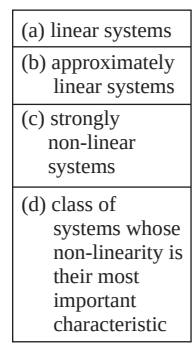


Figure 7.1 A loose classification of systems in terms of linearity/non-linearity

Systems in class (c) will often be linearised to allow certain types of controller synthesis to be carried out. Checks by numerical simulation of the complete unapproximated system plus controller will then be used to determine whether the designs (based on linearised approximation) will be sufficiently valid in practice over a choice of envisaged operating conditions.

Systems in class (d) have their behaviour dominated by non-linearity. Such systems include:

(i) *Stable oscillators*: Governed by continuous non-linear differential equations such as the van der Pol equation. This type of equation exhibits, for the right choice of parameters, limit cycle behaviour. This stable oscillatory behaviour, essentially non-linear in its origins, is very interesting and has been much studied (see Andronov (1973) and van der Pol (1927)).

(ii) *Relay and switched systems*: The systems appear deceptively simple, but, because of the discontinuous non-linearity, special techniques of analysis are required. Because switched systems are both cheap and high performing, they are frequently applied in industry, even in situations for which they are not too well suited (see Tsien (1954)).

(iii) *A variety of systems exhibiting jump resonance, stick-slip motion, backlash and hysteresis*: All of these phenomena can be present as insidious and persistent degraders of performance of control loops (see Gibson (1963)).

7.2 Approaches to the analysis of non-linear systems

As discussed in (ii) above, available methods of analysis are concerned almost entirely with providing stability information.

7.2.1 Lyapunov's second or direct method

This is the only approach that involves no approximation. However, the information produced by application of the method is of limited value for routine system design. For instance, with the aid of the method, a control loop of guaranteed stability may be synthesised. This means that the designed system, if perturbed, will return to equilibrium – maybe in one second, may be in ten-thousand seconds or more. Information on actual performance is totally lacking.

7.2.2 Lyapunov's first method

A beautiful method that depends on local linearisation. It is summarised later in this chapter. Again, the method has little or no design applicability.

7.2.3 Describing function method (described later in this chapter)

This is a linearisation method in which sinusoidal analysis proceeds by the expedient of neglecting harmonics generated by the non-linearities. Thus, the approximation consists in working only with the fundamental of any waveform generated.

The describing function method can be a powerful design tool for a very restricted class of problems.

7.2.4 Sector bound methods

A non-linear function f may be contained within two straight line boundaries. Each of these boundaries is a linear function (Figure 7.2). Envelope methods (a description that is by no means universal) are based on the idea of ensuring system stability in the presence of any and every function that can reside in the envelope. Clearly, the stability results obtained by envelope methods will be sufficient, but not necessary, conditions, since the worst case within the envelope has to be allowed for. Envelope methods are made more interesting by the existence of two famous conjectures. These are as follows:

Aizerman's conjecture: Roughly states, let S be a system containing a non-linearity that can be contained within the linear envelope (Figure 7.2). If, when the non-linearity is replaced by any linear function within the sector as visualised in Figure 7.2, the resulting loop is stable, then the system S is itself stable. Aizerman's conjecture is false, as may be shown by counter-example.

Kalman's conjecture: Roughly states, if a system satisfies Aizerman's conjecture, together with additional reassuring constraints on derivatives, the system S will be stable. Kalman's conjecture is also false, as shown by counter-example (see Leonov (2010)).

It is interesting to speculate on the reasons for the failure of the two conjectures. The easiest line of reasoning, although not necessarily correct, is that harmonics present in the sinusoidal response of the non-linear system have no counterpart in the linear systems that represent the bounds of the approximating sector.

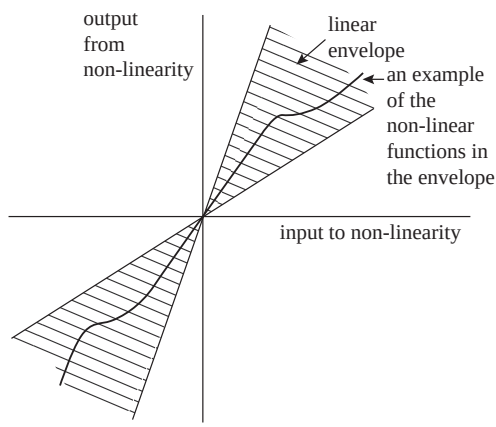


Figure 7.2 A linear envelope that bounds a class of (memoryless) non-linearities

7.3 The describing function method for analysis of control loops containing non-linearities

This method is specifically applicable to a closed loop containing dynamic non-linearities that can be decomposed into a non-linear, non-dynamic block of gain $N(a)$ followed by a linear dynamic block of transfer function $G(s)$ (Figure 7.3). The notation $N(a)$ emphasises that N is an amplitude-dependent gain.

As a simple illustration of the nature of $N(a)$, consider a non-linearity that on receiving a constant input a produces a constant output a^2. We can see that gain, defined as

$$\frac{\text{Output}}{\text{Input}} = \frac{a^2}{a} = a \tag{7.4}$$

Referring to Figure 7.4, we shall assume for linearisation purposes that the output of the block in Figure 7.4(a) is to be approximated as closely as possible by the output of the block in Figure 7.4(b).

For purposes of illustrating the approach of the describing function, we consider a non-linearity f that does not induce a non-zero mean level or cause a phase shift in response to a sinusoidal input. In such a case, the bracketed terms in the output of the block in Figure 7.4(b) disappear and we are left to find the k that causes best agreement between the terms $f(a \sin \omega t)$ and $ka \sin \omega t$. We define the error between these terms as $e(t)$ and then proceed to choose k to minimise the integral of squared error.

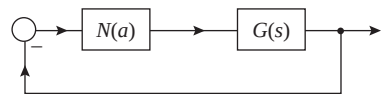


Figure 7.3 The loop containing a linear dynamic system and a non-linear non-dynamic system that is analysed by the describing function method

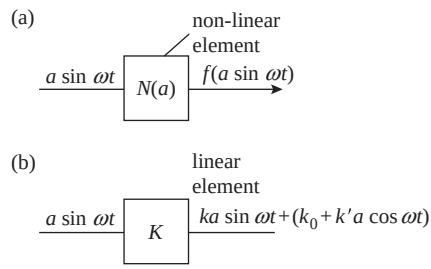


Figure 7.4 (a) A non-linear element; (b) a linear approximation to the non-linear element in (a)

This approach is considered more satisfying than the usual approach of simply neglecting harmonic terms in a Fourier expansion, although the two approaches lead to the same result. Hence, let

$$e(t) = f(a\sin\omega t) - ka\sin\omega t \tag{7.5}$$

we wish to minimise

$$J = \frac{1}{2\pi}\int_0^{2\pi} e(t)^2 d\,\omega t \tag{7.6}$$

substitute for e and differentiate

$$\begin{aligned}
&\frac{\delta J}{\delta k} = \frac{2}{2\pi}\int_0^{2\pi}[f(a\sin\omega t) - ka\sin\omega t](-a\sin\omega t)d\,\omega t\\
&\frac{1}{\pi}\int_0^{2\pi} ka^2\sin^2\omega t\,d\,\omega t = \frac{1}{\pi}\int_0^{2\pi} a\sin\omega t f(\mathrm{a}\sin\omega\mathrm{t})d\,\omega\mathrm{t}\\
&\frac{1}{\pi}ka^2\left(\frac{\omega t}{2} - \frac{\sin 2\omega t}{4}\right)\Big|_0^{2\pi} = \frac{1}{\pi}\int_0^{2\pi} a\sin\omega t f(\mathrm{a}\sin\omega\mathrm{t})d\,\omega\mathrm{t}\\
&\frac{ka\pi}{\pi} = ka = \frac{1}{\pi}\int_0^{2\pi} a\sin\omega t f(a\sin\omega t)d\,\omega t
\end{aligned} \tag{7.7}$$

finally,

$$k = \frac{1}{\pi}\int_0^{2\pi} \sin\omega t f(a\sin\omega t)d\,\omega t \tag{7.8}$$

k can be seen to be the first term in the Fourier expansion of the output of the non-linear block of Figure 7.4(a).

To see how the describing function method develops from this point onward, see Gasparyan (2008).

However, it can be said that, briefly, the further development consists in deriving two loci, one for the non-linear element $N(a)$ (which, recall, has no dynamics) and one for the dynamic element $G(s)$ (which, by definition, is linear). The first locus is a function of amplitude (a) only, while the second is a function of frequency (ω) only.

Especially interesting is the point or points where $G(j\omega)N(a) = -1$, because at such points, there is potentially continuous oscillation around the closed loop. Such points are revealed by plotting loci of $G(j\omega)$ and $-1/N(a)$ in the same complex plane and seeking their points of intersection.

The describing function method allows the user to know whether stable oscillations will occur at an intersection of loci (i.e. that the system is 'attracted' to such points) or whether it is 'repelled' from them.

7.4 Linear second-order systems in the state plane

Note that the name phase plane is used for the special case where (see below) x_2 is the derivative of x_1.

Every linear second-order system with zero input can be expressed in the form

$$\dot{x}_1 = a_{11}\dot{x}_1 + a_{12}x_2 \tag{7.9}$$

$$\dot{x}_2 = a_{21}\dot{x}_1 + a_{22}x_2 \tag{7.10}$$

where the x_i are state variables and the a_{ij} are numerical coefficients, or $\dot{x} = \boldsymbol{A}x$ where x and $\boldsymbol{A}$ are defined by the equivalence between the two representations.

The system has one critical point, where $x = 0$. This point is always the origin (0, 0).

A graph of x_2 against x_1 is called the state plane (Figure 7.5). Solutions of the equation

$$\dot{x} = Ax, \quad x(0) = x_0 \tag{7.11}$$

plotted in the state plane, with time as a parameter along them, are called trajectories. A state plane supplemented by representative trajectories is called a state portrait. The trajectories of a stable system reach or approach the origin of state space with increasing time. Conversely, the trajectories of an unstable system start from the origin and move outwards from it with increasing time.

If the matrix A has two real and distinct eigenvectors, then these eigenvectors are important fundamental trajectories and every solution that is not an eigenvector is a weighted sum of both eigenvectors. The rate of movement of a solution along an eigenvector depends on the magnitude of the associated eigenvalue.

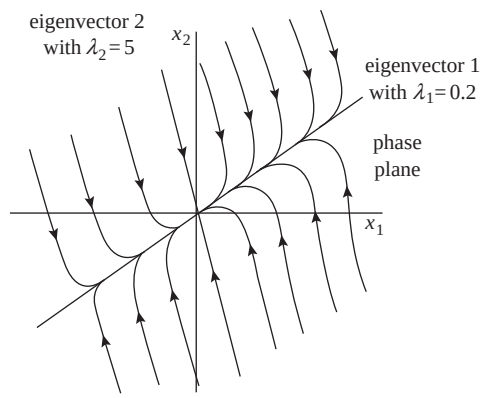


Figure 7.5 The state plane diagram for a second-order linear system with two real negative eigenvalues λ_1, λ_2

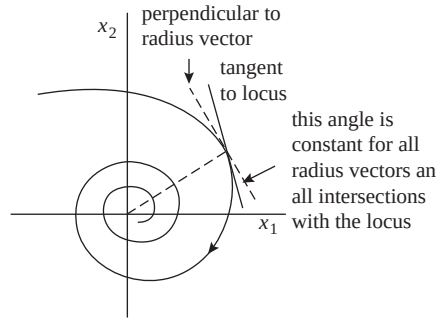


Figure 7.6 The state plane diagram for a second-order system with complex eigenvalues

(An eigenvalue of large magnitude implies rapid movement of the solution along the eigenvector.) All of these points are illustrated in Figure 7.5.

If the matrix A has complex eigenvalues, then the solution is an expanding spiral if the real part of the eigenvalues is positive, and a shrinking spiral if the real part of the eigenvalues is negative. All the spirals are equiangular spirals – i.e. the spirals move outwards or inwards at a constant angle – measured against a rotating vector centred at the origin. These points are illustrated in Figure 7.6. Thus, the global behaviour of a linear second-order system may be characterised by the eigenvalues and eigenvectors of the system matrix A.

7.5 Non-linear second-order systems in the state plane

Consider the set of non-linear second-order systems that can be written in the form,

$$\dot{x}_1 = f_1(x_1, x_2)$$

$$\dot{x}_2 = f_2(x_1, x_2) \tag{7.12}$$

where f_1 and f_2 are differentiable functions.

The system has a number of critical points, given by solving the equations

$$f_1(x_1, x_2) = f_2(x_1, x_2) = 0 \tag{7.13}$$

Let these points be denoted $c_1, c_2, \ldots, c_n$.

The equations may be linearised (see Section 7.8 onward for a description of the procedures involved) to produce the A matrix with typical element

$$a_{ij} = \frac{\partial f_i}{\partial f_i} \tag{7.14}$$

By substituting the co-ordinates of the separate critical points into the general expression for the A matrix, we produce n, generally different, A matrices, $Ac_1, \ldots, Ac_n$.

Now in a small region around each of the critical points, the actual system behaviour is governed by the eigenvalues and eigenvectors of the appropriate A matrix. Thus, the behaviour of the non-linear system in the immediate behaviour of critical points may easily be determined, and for many, but not all, non-linear systems, a phase portrait of the complete behaviour may easily be approximately constructed by continuing the solutions found around each critical point until they join together in a feasible way. (A few numerical solutions of the original non-linear equations can serve to check on the behaviour of any particular trajectory.) A simple example illustrates all these points.

Example The non-linear equation is

$$\dot{x}_1 = x_2 \tag{7.15}$$

$$\dot{x}_2 = -x_1 - x_1^2 - x_2 \tag{7.16}$$

Critical points are (0, 0) and (−1, 0).

The A matrix is

$$A = \begin{pmatrix} 0 & 1 \\ -1 - 2x_1 & -1 \end{pmatrix} \tag{7.17}$$

so that

$$A_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad A_{(-1,0)} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \tag{7.18}$$

$A_{(0,0)}$ has complex eigenvalues with negative real part. $A_{(-1,0)}$ has real eigenvalues +1.08 and –2.08 with associated eigenvectors

$$\begin{pmatrix} 1 \\ 0.618 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1.618 \end{pmatrix} \tag{7.19}$$

The local behaviour around the two critical points is therefore found to be as in Figure 7.7(a) and the feasible state portrait obtained by continuation and joining of trajectories is shown in Figure 7.7(b).

7.6 Process non-linearity: large signal problems

Consider the operations of

(i) accelerating a load using an electric motor;
(ii) heating a block of metal in a furnace;
(iii) growing a population of micro-organisms;
(iv) filling a vessel with liquid.

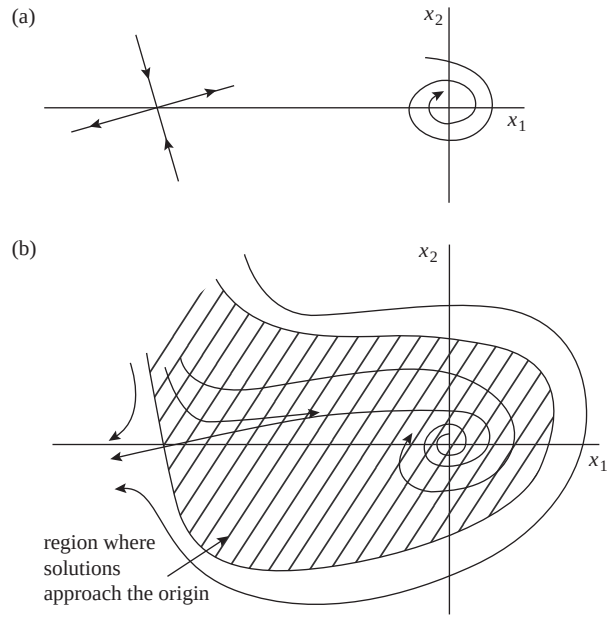


Figure 7.7 (a) The nature of the two critical points of the equation $\dot{x}_2 = -x_1 - x_1^2 - x_2$*; (b) the 'feasible' state portrait for the equation used in (a)*

Each operation has upper limits on its achievable rate of change. In every case, the upper limits are set by rather basic aspects of the design, and the upper limits can only be increased by fairly fundamental redesign of the operations.

Linear control theory (by definition) knows nothing about these limiting factors. Therefore, we may arrange for the limits to be so high that they are never encountered. The process then appears linear but possibly at a high cost in equipment. A more usual approach is to design on linear assumptions although knowing that upper excursions of signals will be sometimes affected by non-linearities. Such an approach needs to be followed by an assessment of the effect on overall performance of the non-linearities. (Such an assessment can be undertaken by either deterministic or stochastic simulations.)

7.7 Process non-linearity: small signal problems

Consider (school physics for once comes in useful) a wooden block at position x on a rough level surface. A small force f is applied where shown (Figure 7.8) and f is gradually increased until, when $f > f_s$ (Figure 7.9), the block suddenly accelerates away. It is now clear that the block will either not move at all (if $f < f_s$) or, if $f > f_s$, move by some minimum amount. In accurate positioning control systems, stiction, for instance in bearings, causes precisely the same difficulty, i.e. there is a

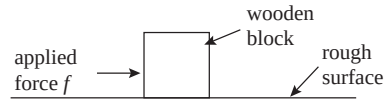


Figure 7.8 A block of wood on a rough surface

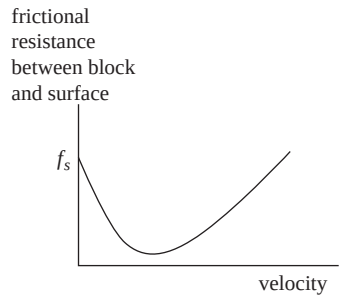


Figure 7.9 The supposed friction characterisation between block and surface in Figure 7.9

minimum unavoidable distance that a shaft must move from rest, if it is to move at all. This phenomenon is sometimes referred to as stick-slip motion.

Other types of small signal non-linearity occur in gear trains.

Considering large- and small-scale linearities simultaneously, it does emerge that, quite often, a high-performance requirement will necessitate the purchase of equipment that is linear across a very wide signal range. Such equipment is very expensive, and, sadly, we cannot usually obtain high performance by attaching a clever control system to a cheap process that has only a narrow range of linear operation.

7.8 Linearisation

7.8.1 The motivation for linearisation

The most powerful tools for analysis and design of control systems operate only on linear models. It is therefore, potentially, very attractive when undertaking the design of a controller for a non-linear system to replace the non-linear system model by a linear approximation.

Questions that arise next are as follows:

- What is meant by linearisation?
- How is it undertaken?
- To what extent are designs, produced using linear approximations, valid in practice when applied to the original non-linear system?

7.9 What is linearisation?

7.9.1 An initial trivial example

The volume V of a sphere is given by

$$V = \frac{4\pi r^3}{3} \tag{7.20}$$

where r is the radius of the sphere.

Suppose $r_0 = 10$, then $V = 4188.79$.
Suppose $r_1 = 10.1$, then $V = 4315.7147$.
Suppose $r_2 = 11$, then $V = 5575.27956$.

These are the full solutions of the non-linear equation for three different r values.

To linearise the equation we operate as follows: Let $V = V_0 + \delta v$, $r = r_0 + \delta r$, then

$$\begin{aligned} V_0 + \delta V &= \frac{4\pi(r_0 + \delta r)^3}{3} \\ &= \left(\frac{4}{3}\right)\pi(r_0^3 + 3r_0^2\delta r + 3r_0\delta r^2 + \delta r^3) \end{aligned} \tag{7.21}$$

while from earlier

$$V = \frac{4\pi r_0^3}{3} \tag{7.22}$$

Subtracting the last equation from the one above it yields

$$\delta v = \frac{4}{3}\pi(3r_0^2\delta r + 3r_0\delta r^2 + \delta r^3) \tag{7.23}$$

Linearisation consists in neglecting terms in δr_2, δr_3, etc., i.e.

$$\delta v = \frac{4}{3}\pi r_0^2\delta r \tag{7.24}$$

and this result could have been obtained directly by using

$$\frac{dv}{dr} = \frac{4}{3}\pi r_0^2 \cong \frac{\delta v}{\delta r} \tag{7.25}$$

To complete this simple illustration, we will see how good the approximations are for two cases, keeping $r_0 = 10$:

(i) When $r_1 = 10{\cdot}1$, $\delta r = 0.1$, $\delta V = 4\pi(10)^2 0.1 = 125.6637$ yielding $V_1 = V_0 + \delta V = 4314.45$ (true solution $= 4315.7147$).

(ii) When $r_2 = 11$, $\delta r = 1$, $\delta V = 4\pi(10)^2 1 = 125.66$ yielding $V_2 = V_0 + \delta V = 5445.28$ (true solution $= 5575.28$).

Clearly, as the perturbation (in this case δr) moves further from the point about which linearisation is performed (in this case r_0), the approximation becomes less valid.

7.9.2 Comments

Thus, linearisation (which we shall discuss in more depth below)

(a) amounts to a local approximation of differentiable functions by derivatives;
(b) is only valid for small perturbations (with small being dependent on context).

However, this is a point of considerable practical importance, we can overcome problem (b) to a considerable extent by linearising a function, not about some constant value (Figure 7.10(a)) but rather about a nominal solution that is expected to be followed approximately (Figure 7.10(b)).

An interesting side-question now arises. Suppose that the linearised equation is itself generating the solution about which successive linearisations are being performed (Figure 7.10(c)). If the perturbations are too large, the accuracy of the

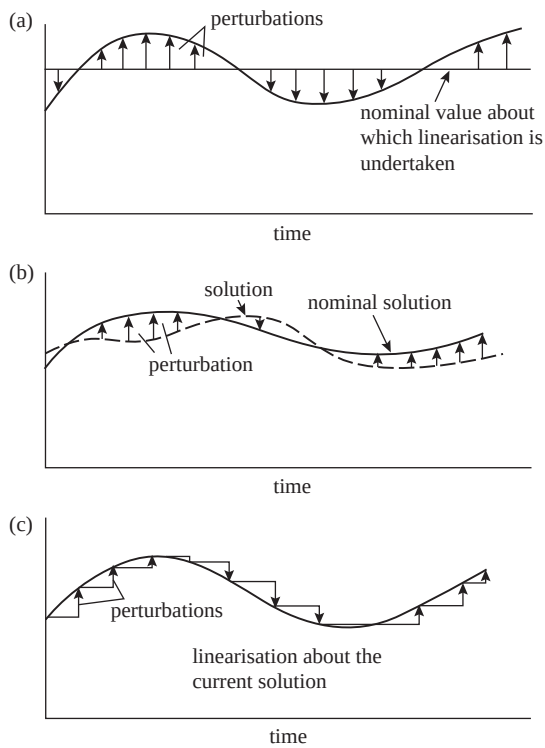


Figure 7.10 (a) Linearisation about a constant value; (b) linearisation about a nominal solution; (c) linearisation about the current solution

linearisation will be poor, and the generated solution will be invalid and the errors cumulative, so that the whole approach will fail. This leads to the topic of numerical solution of differential equations, where, in general, it is not found efficient to use linearisation but rather to use several more terms (say 4) of the Taylor series approximation of a non-linear function to produce the Runge–Kutta approach to numerical solution.

7.10 Linearisation about a nominal trajectory: illustration

Let the equation

$$\dot{x} = f(x) + g(u) \tag{7.26}$$

represent a non-linear industrial process that repeats the same routine day after day. Each day it receives a nominal input $u_N(t)$, in response to which it produces a nominal output $X_N(t)$ (Figure 7.11(a)). Linearisation about the nominal trajectories consists in producing the perturbation equation

$$\delta\dot{x} = \frac{\partial f}{\partial x}\Big|_{x=x_N(t)} \partial x + \frac{\partial g}{\partial u}\Big|_{u=u_N(t)} \delta u \tag{7.27}$$

This linear equation models the process behaviour about the nominal trajectories (Figure 7.11(b)).

In practice (for instance if the application is a repetitive batch process), the nominal trajectories will often be taken as the mean of a large number of typical performances. Any individual performance can then be modelled as $x(t) = x_N(t) +$ the solution of the perturbation equation.

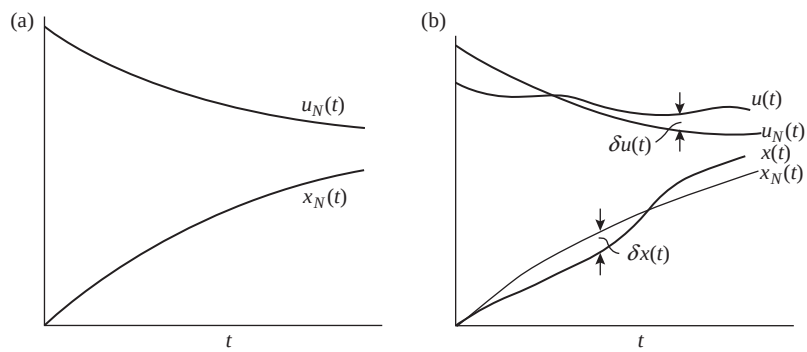


Figure 7.11 (a) The nominal input uN(t) provokes the nominal response xN(t); (b) perturbation about the nominal trajectories

7.11 The derivative as best linear approximation

We can, if we wish, define the derivative of a function $f(x)$ as the unique linear function $df|_x$ that best approximates f near to x (Figure 7.12).

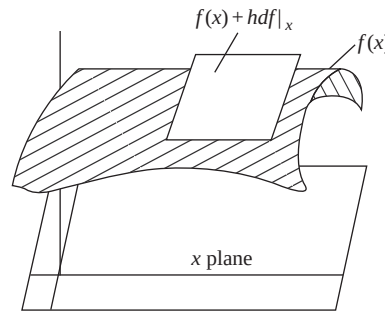


Figure 7.12 The derivative approximates the function f locally by the tangent plane shown

In the usual system of co-ordinates, the linear transformation df has the matrix

$$F = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial f_n}{\partial x_1} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{bmatrix} \tag{7.28}$$

which is called the Jacobian matrix of f at x.

The goodness of the approximation depends on $df|_x$. If $df|_x$ is non-zero then, in general, the approximation is good. To understand this, look at the approximation of sin x at two values $x = 0$ and $x = \pi/2$ as sketched in interlude 7A.

7A The inverse function theorem

The inverse function theorem gives an interesting view of approximation. It says that, if the derivative df of f at x has an inverse then so does f locally, i.e. in some region U in x there exists a function g such that

$$g(f)(x) = x \text{ for all } x \text{ in } U$$
$$f(g)(y) = y \text{ for all } y \text{ in } V$$

i.e. f has an inverse, g, on the restricted regions U, V.

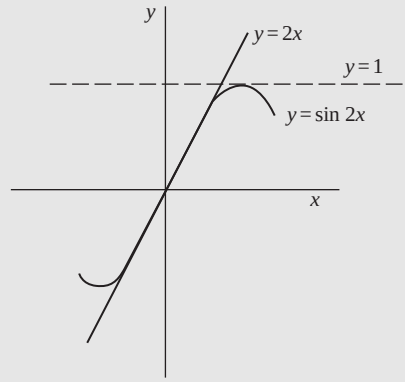


Figure 7.13 The curve $y = \sin 2x$ is well approximated by its first derivative $y = 2x$ at $x = 0$. At $x = \pi/8$ we have as linear approximation $y = \sin \pi/4 + 0 = 1$, a poor approximation

Within the regions U, V we can replace the x co-ordinates by the corresponding y co-ordinates (see Poston and Stewart (1976), p. 9) and then over the region U the function f is completely linearised without approximation. However, if df is not invertible (tested by checking for singularity of Jacobian matrix), then such approximation is not possible. Overall, the following result holds: If f has a non-zero gradient at x then we can find a smooth change of co-ordinates in some ball U around x by which the expression of f on u becomes linear.

Where the gradient is zero, the Jacobian is, by definition, zero and approximation has to be carried out by relying on the matrix of second derivatives, i.e. on the Hessian matrix H.

As can be seen in Figure 7.13, the non-linear function $\sin x$ can be well approximated at $x = 0$ (by the linearisation $y = 2x$), but at $x = \pi/8$, the linear approximation $y = 1$ is poor because the Jacobian is zero there.

7B The concept of transversality

When a line pierces a plane a slight variation in either the line or the plane will not affect the nature of the intersection.

However, if a line touches a plane tangentially, then slight variations will affect the nature of the meeting, resulting in, e.g. two piercings of the plane, or no meeting with the plane at all (Figure 7.14). These ideas, which are closely connected with catastrophe theory, have obvious connections with robustness as defined in terms of insensitivity to parameter changes.

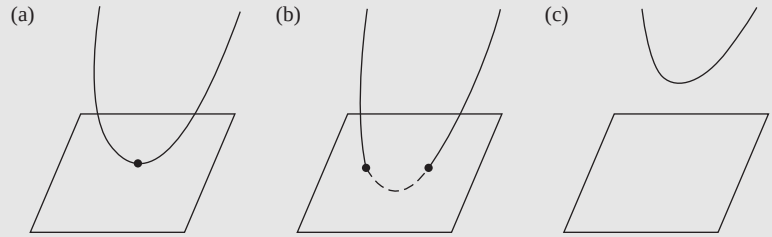


Figure 7.14 *(a) As a typical situation in three-dimensional space – a loop touches a place tangentially. (b, c) Typical situations in three-dimensional space – a line (b) pierces the plane in two places, (c) fails to meet the plane*

7C The forced pendulum: a simple example of chaotic behaviour

The linearised equation of a simple undamped pendulum is

$$\theta + \frac{g}{l}\theta = \theta + \omega_n^2\theta = 0 \tag{7.29}$$

where, as usual, θ represents the angular deflection from equilibrium, l is the length of the pendulum, g is the gravitational acceleration constant for the location and

$$\omega_n = \sqrt{\frac{g}{l}} \tag{7.30}$$

is the natural frequency of the system. Setting $\omega_n = K = 1$ (K is brought in as an intermediary to remind us that ω_n will no longer be the constant natural frequency for the enhanced system we are about to introduce.

We now move to the enhanced, forced, unlinearised equation

$$\theta'' + \frac{g}{l}\sin\theta = \cos\omega t \tag{7.31}$$

or

$$\theta'' + K\sin\theta = \cos\omega t \tag{7.32}$$

and setting $K = \omega = 1$

$$\theta'' + \sin\theta = \cos(t) \tag{7.33}$$

We note several features of the above equation:

The magnitude (unity) of the forcing term is very large and will presumably soon take the system out of the linear range regardless of the chosen initial conditions. The forcing frequency is the same as the natural frequency

of the linearised system. The restoring force of the non-linear system will be significantly less than that for the idealised linear system for every displacement θ since, for any θ,

$$\left(\frac{|\sin\theta|}{|\theta|}\right) \leq 1 \text{ and } \left(\frac{\sin(1)}{1}\right) = 0.84 \tag{7.34}$$

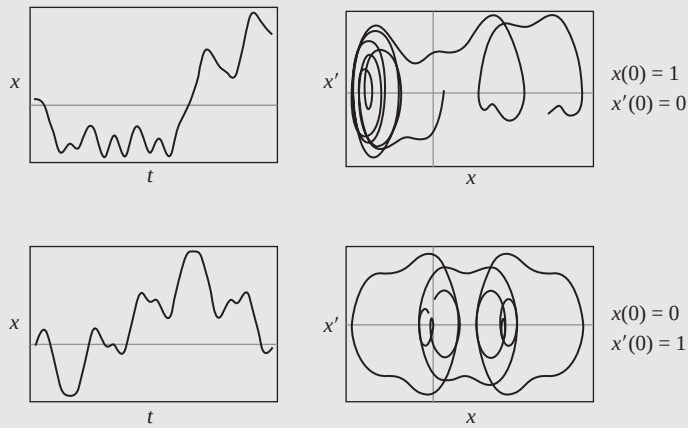


Figure 7.15 Time responses for the equation $\theta'' + \sin\theta = \cos(t)$ for two different sets of initial conditions

The time responses for the equation

$$\theta'' + \sin\theta = \cos(t) \tag{7.35}$$

for two different sets of initial conditions are sketched in Figure 7.15. Perhaps the right-hand (phase plane) sketches are the most illuminating and although this is a casual exploratory analysis, the chaotic interactions between the forcing term and the inherent natural periodic behaviour are already indicated.

A final small lesson to be learned from this example: In the right-hand phase-plane diagrams, the trajectories are seen to cross. Such crossing of trajectories is seldom seen in phase-plane diagrams, because in autonomous (unforced) systems such crossing is not possible. Here, trajectory crossing is possible because of the presence of the periodic forcing term, meaning that, from the same point in phase space, different future trajectories can evolve at different times.

Chapter 8

Limits to performance

Most closed loop systems become unstable as gains are increased in attempts to achieve high performance. It is therefore correct to regard stability considerations as forming a rather general upper limit to control system performance. Also, as will be discussed in this chapter, achievable rates of change are always constrained in practice by equipment limitations.

8.1 Stability: initial discussion

A stable system is one that, when perturbed from an equilibrium state, will tend to return to that equilibrium state. Conversely, an unstable system is one that, when perturbed from equilibrium, will deviate further, moving off with ever-increasing deviation (linear system) or possibly moving towards a different equilibrium state (non-linear system) (Figure 8.1).

All usable dynamical systems are necessarily stable – either they are inherently stable or they have been made stable by active control techniques. For example, a ship should ride stably with its deck horizontal and tend to return to that position after being perturbed by wind and waves (Figure 8.2).

Stability occupies a key position in control theory for the reason that the upper limit of the performance of a feedback control system is often set by stability considerations, although most practical designs will be well away from the stability limit to avoid excessively oscillatory responses.

It is possible to check whether a system is stable or not by examining the behaviour with time, following an initial perturbation (Figure 8.3). To establish whether a system is stable or not, we do not need to know the solution of the system equations, but only to know whether after perturbation the solution decays or grows.

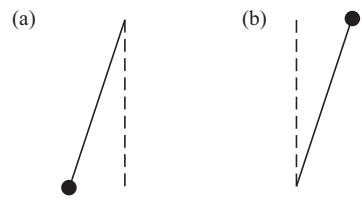


Figure 8.1 (a) Stable system; (b) unstable system

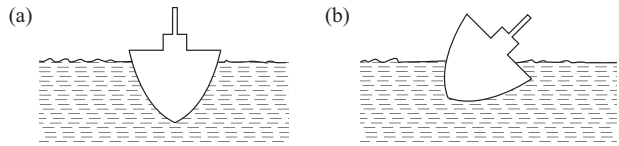


Figure 8.2 (a) Equilibrium position of ship; (b) ship when perturbed tends to equilibrium

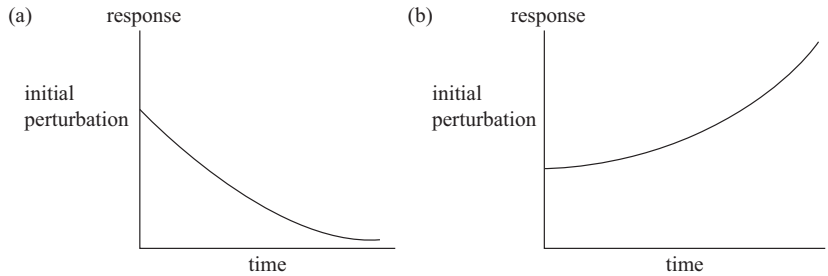


Figure 8.3 (a) Response of a stable system after perturbation; (b) response of an unstable system after perturbation

Notice that, for a linear system, the responses to initial perturbations of different magnitudes are identical except for a scaling factor. That is, let x_0 be the initial perturbation and $x(t)$ the resulting response, then the response to a perturbation kx_0 will be $kx(t)$. Therefore, if a system is stable in response to one magnitude of perturbation, it will be stable in response to all other magnitudes.

8A Stability theory: a long-term thread that binds

Stability analysis has a long and honourable history providing a thread that pre-dated control theory and then linked in with it.

Stability studies were applied to problems in planetary motion before control was even considered and most famously to the problem of the nature of Saturn's rings (Figure 8.4), for which Maxwell was awarded the Adams Prize. (Maxwell conjectured correctly that for the rings to be stable they must be particulate.)

I took the 'top' example in Figure 8.5 from an examination paper that Maxwell set for an undergraduate class at King's College, London. It is not recorded how many, if any, answered the question with any degree of success, but at the time no suitable stability criterion existed and the student would need to invent one. To make the question even more demanding, Maxwell added a rider to his question asking the student to state whether any invented stability criterion in the solution was necessary and sufficient!

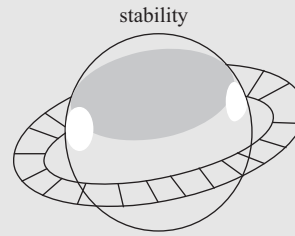
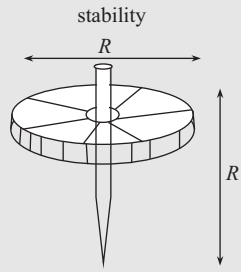


Figure 8.4 Saturn's rings. Maxwell's Adams Prize essay showed the rings to be particulate



The disc of a spinning top can be moved to three possible positions:
(i) in the top 1/3 of the rod
(ii) in the centre 1/3
(iii) in the lower 1/3.
Show that the device will operate stably as a top only in positions (i) and (iii)

Figure 8.5 Maxwell's exam question to King's College London

The Hurwitz, Routh and similar criteria (see interlude 8B) require knowledge of the differential equation of the system that is to be analysed.

Lyapunov's two powerful theorems (Section 7.2) have both algebraic and geometric interpretations that have allowed them to link with many aspects of non-linear control.

The Nyquist and Bode criteria that came next in the development require knowledge only of frequency responses in graphical form. These can be obtained experimentally and can form the basis for synthesis of controllers that will yield desired stability margins. This development allowed the earliest robust control systems to be systematically designed.

8.2 Stability for control systems: how it is quantified

Let Σ be a linear system that is in an initial condition x_0 at time t_0, then the state of the system for $t > t_0$ is given by an equation of the form

$$x(t) = Ae^{\alpha t} + Be^{\alpha t} + \cdots \tag{8.1}$$

where the number of terms depends on the dynamic complexity of the system, and A, B, ... terms depend only on the initial condition x_0, and where the exponents α, β, ... depend on the parameters of the system.

In general, the exponents α, β, ... are complex and it is clear that, if even one of the exponents has a positive real part, then part of the solution of $x(t)$ will increase without bound as t increases and the system is seen to be unstable (since $e^{\alpha t} \to \infty$ as $t \to \infty$ if the real part of α is positive).

Stability therefore is governed only by the real parts of the exponents $\alpha, \beta, \ldots$. If our main concern is with stability, we therefore look in detail at these exponents. Let the dynamic systems have the mathematical model $H(s) = P(s)/Q(s)$. Then the exponents are the solution of the equation $Q(s) = 0$ (the auxiliary equation). These exponents are also called the poles of $H(s)$. Solutions of the equation $P(s) = 0$ are called the zeros of $H(s)$. It is useful to plot the poles and zeros of a system in the complex plane. Poles (marked X) and zeros (marked 0) appear always on the real axis or in complex conjugate pairs, arranged symmetrically above and below the real axis.

Recalling that if any exponent (pole) has a positive real part then the system is unstable, we can see that if any pole is in the right half of the pole–zero diagram then the system Σ is unstable and this is a major stability test for a system describable by a transfer function $G(s)$.

Therefore, the solution yielded by a system after perturbation is governed by the roots of its auxiliary equation if the system model is a transfer function, and by the roots of the characteristic equation (i.e. by the eigenvalues) if the model is a matrix. The situation is summarised in Table 8.2.

Control theory uses the stability tests indicated in Table 8.2 to yield qualitative stability information ('the system is stable' or 'the system is unstable') from

Table 8.1 Some milestones in stability theory

A long-term thread that binds
- Maxwell: governors, Saturn's rings, spinning top
- Lyapunov: two stability theorems
- Hurwitz Routh: stability information from the coefficients of the (unsolved) differential equation
- Nyquist: graphical frequency response method
- Bode: developments of Nyquist approach
- Evans: root locus interpretation of Nyquist approach
- Jury: sampled data formulations
- Doyle: contributions to developing robust control methods

Table 8.2 Stability aspects of system models

System model	**Stability governed by**
Bode diagram	Gain and phase relationship
Differential equation	Roots of auxiliary equation
Transfer function	Poles of transfer function
System matrix	Eigenvalues

differential equations, difference equations, transfer functions or system matrices. Table 8.1 highlights some of the famous names of stability theory.

The three forms of tests in Table 8.2 are all virtually the same test with relabelled variables. They all suffer from the same disadvantage – each test requires the solution of an equation of the form

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s^1 + a_0 = 0 \tag{8.2}$$

In detail, this means finding every complex number α that satisfies

$$\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1 \alpha^1 + a_0 = 0 \tag{8.3}$$

to yield the set of complex numbers $\{\alpha_1, \ldots, \alpha_n\}$ that are the roots required by the stability test that is to be administered.

If in (8.2), $n \leq 2$, the solution follows 'almost by inspection' if $2 < n \leq 4$, then we can use analytic methods (Tartaglia's method for $n = 3$, Ferrari's method for $n = 4$), while if $n > 4$, then by the celebrated proof due to Abel, no analytic solution can exist (see Turnbull (1963) and for a detailed discussion, Burnside and Panton (1892)).

It is, of course, possible to solve any particular equation of any order computationally, provided that it has numerical coefficients throughout. However, in the inevitable iterations of a systems design project, it is very useful to be able to work, at least partially, with as yet unassigned coefficients.

Thus, for $n > 4$, it would be extremely useful to be able to answer the question (applied to (8.2) and using only a knowledge of the coefficients $[\alpha_i]$): In what region of the complex plane do the roots $[\alpha_i]$ lie?

8B The ingenious method of Hurwitz

One solution to the problem came about as follows. The engineer A.B. Stodola, working on the dynamics of water-driven turbine systems, had been able already in 1893 to solve the stability problems that arose from his highly approximated model of order 3 ($n = 3$ in our (8.2)). Although he was not in a direct position to apply the tests outlined in our Table 8.2 (not yet invented), he was equivalently able to apply the known work of Maxwell (1868) on systems of that order.

However, when Stodola produced a more complete model, with fewer approximations, for his turbine systems, he encountered the same problems that are described here. In modern terms, he wanted to know the location of the roots α_i of (8.2) from a knowledge of the coefficients a_i. The mathematician A. Hurwitz, working at the same institution (ETH Zurich) as Stodola, produced the Hurwitz criterion to solve precisely this problem. Stodola was able immediately to apply the criterion to ensure the stability of the design for a new hydroelectric power station that was being built at Davos.

Almost simultaneously, and independently, the Cambridge mathematician E.J. Routh developed an equivalent test, now called the Routh array test, to achieve exactly the same result as the Hurwitz criterion. Many control engineering texts explain one or other of the tests and with loose terminology indeed refer to it as the Routh–Hurwitz criterion.

Notice carefully that the Hurwitz criterion and the Routh array test apply to differential equations and hence also to the transfer functions and A matrices corresponding to such differential equations. They cannot be used to determine the stability properties of difference equations, since for difference equations a different question has to be asked; i.e. are all the roots α_i inside the unit circle in the complex plane? Equivalent to the Hurwitz test for differential equations is the Jury test for difference equations. (See Kuo (2009) for details of the Jury test.) Unfortunately, Jury's test can be unwieldy and this writer finds the so-called w transformation method preferable. In this method, the difference equation is transformed into a differential equation that has the same stability properties. The differential equation, obtained by transformation, is then tested as usual by (say) the Hurwitz method.

8.3 Linear system stability tests

Table 8.3 summarises the stability tests that are available for linear systems. Frequency response methods are widely used to synthesise closed loop systems having predetermined stability characteristics (refer back to Chapter 4).

Table 8.3 Linear system stability tests

System description	Recommended stability test
Continuous time systems	
Differential equations	Roots of auxiliary equation
Transfer functions	Poles
System matrices	Eigenvalues
	(Apply Hurwitz or Routh criterion)
Discrete time systems	
Difference equations	Roots of auxiliary equation
Transfer functions	Poles
System matrices	Eigenvalues
	(Jury test or w transformation then Hurwitz test)

8.4 Stability margin

From what has already been said, it can be inferred that there is a boundary between stable and unstable systems. A usable system must not only be stable but it must be away from the boundary of instability by some sufficient safety margin.

8.5 Stability tests for non-linear systems

Why stability testing of non-linear systems is difficult

For a linear system, all solutions are 'topologically similar'. For instance (Figure 8.6), for a linear system, all responses to initial perturbations of different magnitudes are similar (in a geometric sense). Thus, if an initial perturbation $p(0)$ causes a response $x(t)$, then a scaled-up perturbation $kp(0)$ will cause a scaled-up response $kx(t)$.

However, the behaviour of a non-linear system can exhibit many surprising features. For instance, it is easy to synthesise a non-linear system whose response to two different initial perturbations $p_1(0)$, $p_2(0)$ is as shown in Figure 8.7.

It should be immediately obvious that even the definition of stability for a non-linear system will need to be carefully thought out.

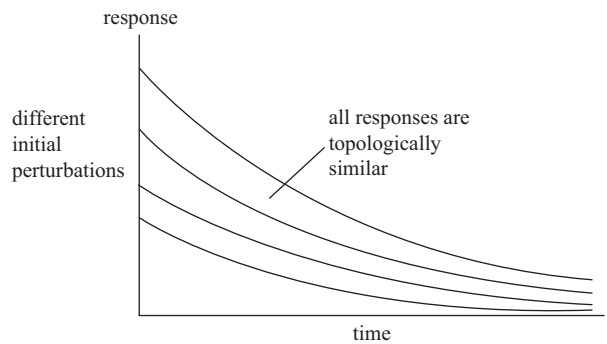


Figure 8.6 The family of responses to perturbations of different magnitudes for a linear system

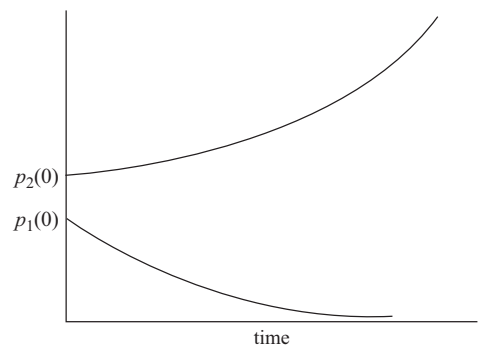


Figure 8.7 It is possible for a non-linear system to be stable for a perturbation $p_1(0)$ while being unstable for the perturbation $p_2(0)$

8.6 Local and global stability

In this treatment we consider non-linear differential equations and operate in the phase plane, thus effectively limiting illustrations, although not results, to second-order systems. (We note in passing that non-linear differential equations do not yield transfer functions, poles, matrices, eigenvalues, frequency response descriptions, superimposable time responses or decomposable time solutions – i.e. auxiliary equations and complementary functions.)

The response to an initial perturbation as in Figure 8.8(a) can also be shown in the phase plane as Figure 8.8(b), where time is a parameter along the trajectory.

A non-linear system where solutions starting at all points in the phase plane tend to the origin will be called globally stable – we can imagine that the origin is an attractor of solutions and that the domain of attraction is the whole of the phase plane.

In the case when the domain of attraction of the origin is a finite region in the phase plane, we call the system locally stable around the origin (Figure 8.9).

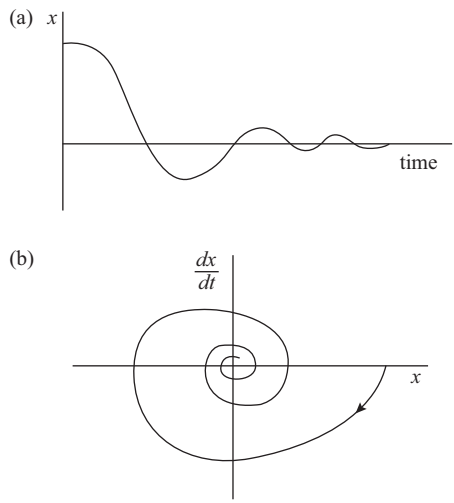


Figure 8.8 (a) A time response; (b) the same response plotted in the phase plane

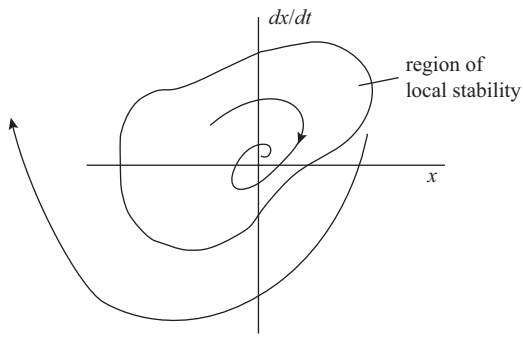


Figure 8.9 A region of local stability in the phase plane

8.7 Lyapunov's second (direct) method for stability determination

Lyapunov's second method (often equivalently referred to as his direct method) has the following properties:

(i) It can be understood most rapidly by reference to the energy contained in a system and the rate of change of that energy.
(ii) Notwithstanding (i), it can be applied to abstract mathematical systems in which energy cannot be defined.
(iii) It has a very valuable geometric interpretation.

We can bring point (i) to life by noting that a moving railway train whose brakes are applied will come to rest when its kinetic energy has all been dissipated in the brakes. If we wanted to calculate the stopping distance of such a train, it is possible to imagine using a method based on energy and its rate of change. Moving to a second viewpoint, it is obvious that the ball in a cup is at a point of minimum potential energy, whereas the ball on a dome is at a point of maximum potential energy (Figure 8.10). The relation between the energy minimum/maximum and the stability/instability of the balls is no accident.

The geometric interpretation of Lyapunov's second method is that 'a system is stable to the origin provided that every closed contour described by the so-called Lyanupov V function is always penetrated from outside to inside by solution trajectories of the differential equation and never in the reverse direction (Figure 8.11).

Figure 8.10 (a) Ball in a cup; (b) ball on a dome

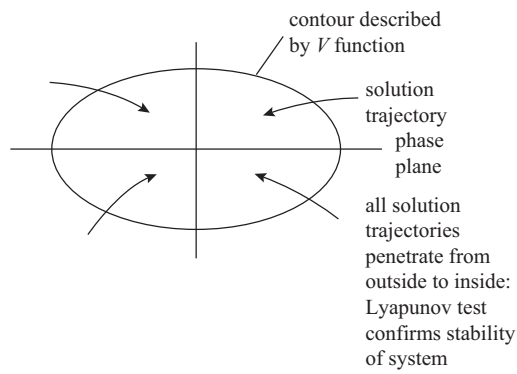


Figure 8.11 All solutions penetrate the V function contour from outside to inside

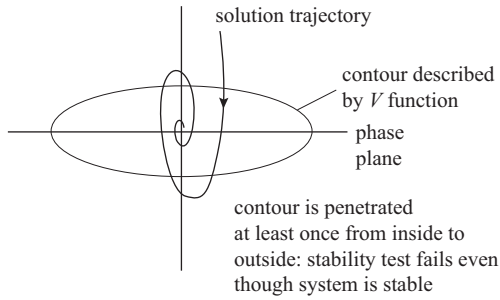


Figure 8.12 Contour is penetrated from inside to outside – stability test fails

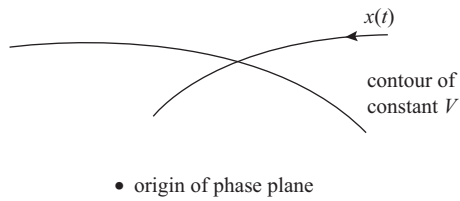


Figure 8.13 A trajectory crosses a contour of the V function

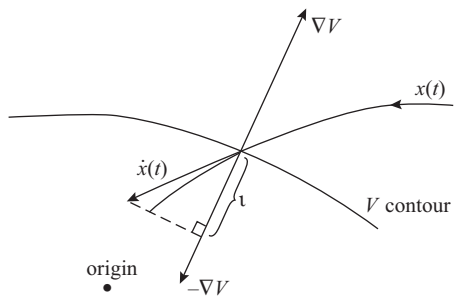


Figure 8.14 Figure 8.13 enhanced by gradient vector and tangent

Notice that some V functions will fail to confirm the stability of some stable systems as illustrated in Figure 8.12.

Lyapunov's test fails because at least one trajectory penetrates from inside to outside. We can see that the Lyanupov test is a sufficient condition for stability – it is not necessary.

8C Geometric interpretation of Lyapunov's second method

Consider a solution trajectory $x(t)$ crossing a contour of constant V on its way towards the origin of the phase plane (Figure 8.13). Let the tangent to $x(t)$ be $\dot{x}(t)$ and let grad V and –grad V be drawn in as shown (Figure 8.14).

Define $1 = \langle \frac{\nabla V}{\|\nabla V\|}, \dot{x} \rangle \langle , \rangle$ indicates inner product i.e. 1 is the projection of x onto the gradient vector ∇V. Note from Figure 8.14 that 1 is a vector, orthogonal to the V contour and that if 1 is negative, pointing towards the origin for every solution $x(t)$ and for every V contour, then the system is stable to the origin within the outermost of the V contours investigated.

Assume that V is positive definite and that lines of constant V form an increasing basin with the origin at its lowest point. Then the usual test that dV/dt must be negative definite for stability to the origin can be seen to be the same as asking that the vector 1 in Figure 8.14 should point inwards. This is so since

$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} = \langle \nabla V, \dot{x} \rangle \tag{8.4}$$

which is the same (except for a scaling factor) as the expression for 1 in the figure.

8.8 What sets the limits on the control performance?

Let $G(s)$ be a model for any process, connected into a control loop with a controller $D(s)$ whose transfer function is under our control. Let the overall model of the loop be represented by $H(s)$ (Figure 8.15).

We ask: For a given $G(s)$, can we, by choice of $D(s)$, synthesise any $H(s)$? The following discussion is a continuation of an earlier discussion in Section 3.1.

From (3.4) and (3.3) (repeated here for convenience), we know that the overall transfer function $H(s)$ of the loop is (3.4)

$$H(s) = \frac{G(s)D(s)}{1 + G(s)D(s)} \tag{8.5}$$

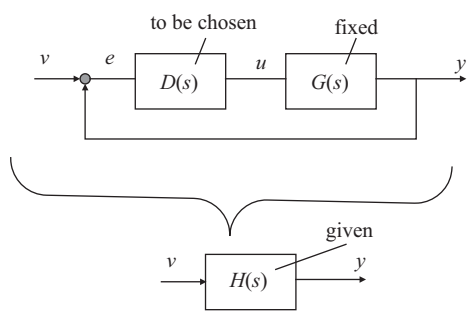


Figure 8.15 Choosing D(s) to achieve a given H(s)

and that the controller $D(s)$ can be chosen using (3.3)

$$D(s) = \frac{H(s)}{G(s)(1 - H(s))} \tag{8.6}$$

As an illustration of an ambitious design, let

$$G(s) = \frac{1}{1 + 1000s} \tag{8.7}$$

i.e. $G(s)$ has a time constant of 1000 s. We ask: Can the controlled system be forced to have a transfer function of

$$H(s) = \frac{1}{1 + s} \tag{8.8}$$

by the connection of a suitable controller, i.e. can the system, when under control, be forced to respond 1000 times faster, with a time constant of 1 s?

This is a generic question of great practical importance: What sets an upper limit on the performance that can be obtained by adding control to a particular process G? The complete answer will not be found by application of control theory but let us continue the example and then discuss the result.

Putting the values into the equation for $D(s)$ yields

$$\begin{aligned} D(s) &= \frac{H(s)}{G(s)(1 - H(s))} = \frac{1/(1 + s)}{(1/(1 + 1000s))(1/(1 - (1/(1 + s))))} \\ &= \frac{1 + 1000s}{s} = \frac{u(s)}{e(s)} \end{aligned} \tag{8.9}$$

or

$$u(s) = \frac{e(s)}{s} + 1000e(s) \tag{8.10}$$

This controller can be realised by the hardware of Figure 8.16.

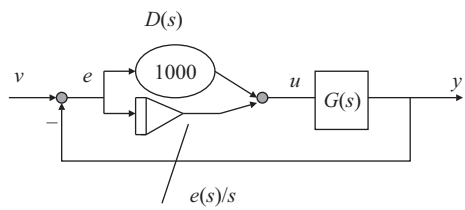


Figure 8.16 A hardware realisation to synthesise the required controller D(s)

Physically there is no reason why the system of Figure 8.16 cannot be built. However, we note that, when the value v is changed suddenly to produce an error $(v - y)$, the output from the controller will instantaneously be 1000 $(v - y)$, which may saturate the actuator of the process $G(s)$ for any significant perturbation of v and, additionally, noise entering the loop may be expected to cause problems. Thus, we conclude that if we are over-ambitious in our attempt to obtain high performance, we may meet limits caused by the finite power rating of signals that the process $G(s)$ can receive.

However, in applications, we frequently do need to work around the loop from small sensor signals whose task is to carry information to the point where a large load of one sort or another may have to be moved, sometimes very rapidly. Such targets are not achievable by using large numerical gains in control loops but rather by power amplification.

To progress, consider particular applications. Imagine a hydroelectric power station where a huge controlled valve varies the flow of water to a set of turbines driving generators to vary the power generated and hence maintain the frequency of the whole supply. Such an application can be found at the Swedish hydro-generating plant at Harsprånget near the Arctic Circle. At 939 MW, produced by 1000 m^3/s of water falling from 107 m, this is Sweden's largest hydroelectric station. Interestingly, although there are quite a number of large hydro plants all connected to the Swedish grid, Harsprånget has the main task of controlling the frequency of the whole grid. A delicate frequency sensor produces a signal of only a few mV and a closed loop system must drive the very large water valve in this application. This is achieved through an increasing sequence of amplifiers, motor generators and finally through a hydraulic actuator (Figure 8.17). This enormous amplification is seen to be stage-wise power amplification and not simply multiplication of gain. Most other applications will meet a maximum rate constraint in

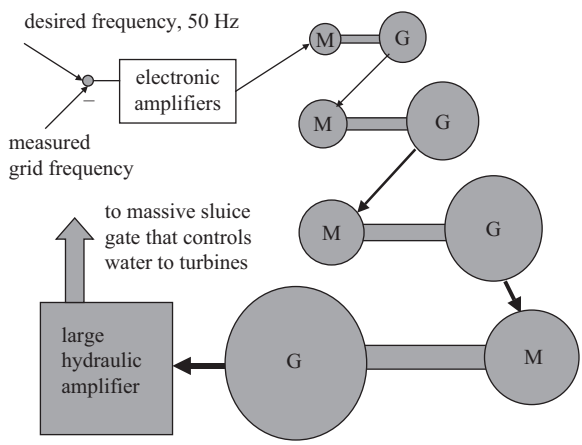


Figure 8.17 How power amplification is obtained in hydro frequency control

the form of the diameter of a pipe, the capacity of a heating burner, the power limitation of a motor or even a biological constraint such as that on the rate of organism growth.

8.9 How robust against changes in the process is a moderately ambitious control loop?

Suppose that a control loop is designed to improve the rate of response of a process $G(s)$ by a factor of ten times. How robust will the resulting loop be against changes in the process? We take a very simple example where

$$G(s) = \frac{1}{1+s} \tag{8.11}$$

and we shall design a controller $D(s)$ such that the resulting closed loop $H(s)$ has the transfer function

$$H(s) = \frac{1}{1+0.1s} = \frac{10}{10+s} \tag{8.12}$$

so that the closed loop system will respond ten times faster than the uncontrolled process. The necessary controller will have the model (see again Figure 8.15)

$$D(s) = \frac{H(s)}{G(s)(1-H(s))} = \frac{10/(10+s)}{(1/(1+s))(1-10/(10+s))} = \frac{10(1+s)}{s} \tag{8.13}$$

This controller in closed loop with the given $G(s)$ will produce the required transfer function $H(s)$.

The purpose of this section is to check the effect of process changes on closed loop performance. We therefore postulate a significant but feasible change in the process time constant to yield the modified process model

$$G'(s) = \frac{1}{1+1.4s} \tag{8.14}$$

and calculate the resulting model, say $H'(s)$, of the closed loop as

$$H'(s) = \frac{(1/(1+1.4s))(10(1+s))}{1+(1/(1+1.4s))(10(1+s)/s)} = \frac{10(1+s)}{1.4s^2+11s+10} \tag{8.15}$$

$H'(s)$ has two real poles at approximately $s = -2$ and $s = -5$ so it is not immediately obvious how the response of $H'(s)$ will differ from that of $H(s)$ (one pole at $s = -10$). To investigate this we shall calculate the step response of $H'(s)$ and compare it with that of $H(s)$.

(As a piece of reinforcement learning, we note that the step response of $H'(s)$ can be found by taking the inverse Laplace transform of $(H'(s)u(s))$, where $u(s) = 1/s$

is the transform of a unit step time function. Alternatively, we can argue that the response of $H'(s)$ to a unit step must be the integral of the impulse response of $H'(s)$. Since in the Laplace domain, the operation of integration is accomplished by multiplication by $1/s$, we again need to introduce this term before inverse transformation. In this short reminder, we have shown that the possibly puzzling fact that $1/s$ is simultaneously the transform of a unit step time function as well as the Laplace domain variable representing integration does not lead to any inconsistency.)

Therefore, the step response of $H'(s)$ as a time function will be found by inverse Laplace transformation as

$$\mathcal{L}^{-1}\left(\frac{1}{s}\frac{10(1-s)}{(1.4s^2+11s+10)}\right) \tag{8.16}$$

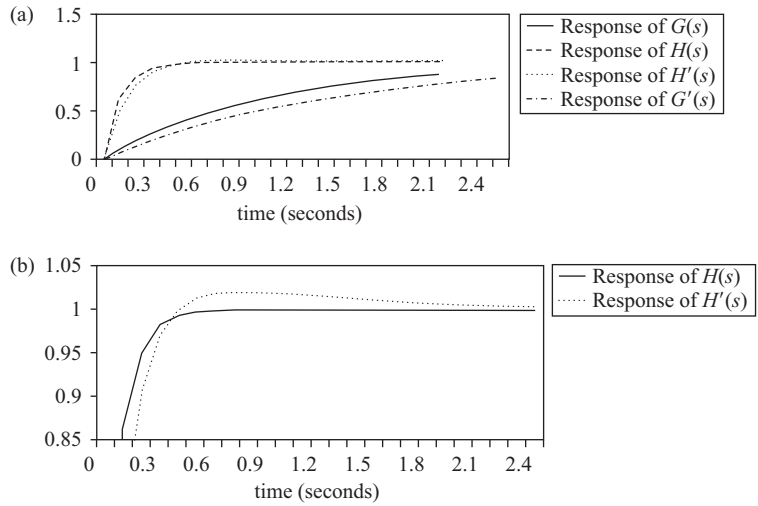


Figure 8.18 (a) The step responses of processes G(s), G′(s) alone and under closed loop control (H(s), H′(s)); (b) detail showing overshoot in response of H′(s)

Figure 8.18(a) shows plots of the step responses of $H(s)$, $H'(s)$ with, for comparison, those of the processes $G(s)$, $G'(s)$.

The response of $H'(s)$ is remarkably close to that of $H(s)$, considering the large change in the process that has taken place. Closer examination (Figure 8.18(b)) shows, however, that the response of $H'(s)$ suffers from an overshoot that decays with a long time constant that is a legacy from the failure of the fixed controller $D(s)$ being unable to cancel the pole of the changed process $G'(s)$.

Overall, though, the result confirms the hoped-for robustness of a single feedback control in the face of process changes.

8.10 Limits and constraints on synthesis: summary of points

Given any process $G(s)$ and any required overall transfer function $H(s)$, it is always possible to calculate a controller $D(s)$ to ensure that the required $H(s)$ is obtained by substitution of $G(s)$ and $H(s)$ into the relevant equation.

Clearly in, say, aircraft design, $G(s)$ could be a model of a low-performance aircraft, $H(s)$ could be the model of a high-performance aircraft and $G(s)$ could be 'turned into' $H(s)$ merely by the addition of a suitable controller $D(s)$. However,

(i) Not every $D(s)$ that can be written down is physically synthesisable.
(ii) Even though $D(s)$ may be synthesisable, a very ambitious choice of $H(s)$ will necessarily lead to signals of large magnitude being generated during transients, necessitating the use of expensive powerful components.
(iii) A very ambitious choice of $H(s)$ may lead to a control system whose performance is excessively sensitive to small changes in the process characteristics.

8.11 Systems that are difficult to control: unstable systems

Unsurprisingly, an inherently unstable system is usually difficult to control. Yet the combination of an inherently unstable aircraft, made usable by active stabilisation and control, is often attractive on grounds of overall efficiency and such a combination is often used in high-performance military aircraft design.

There are also examples of deliberately near-unstable systems in nature. For instance, over many centuries, flying insects have evolved from stable passive long-tailed shapes, able to glide without exercise of brain power, to more efficient, but inherently unstable, short-tailed versions that include fast-acting measurement and closed loop control and stabilisation (Dudley, 2002).

Unstable systems have one or more poles in the right half complex plane and the most obvious control strategy would be to cancel the unstable poles by coincident right-half-plane controller zeros (Figures 8.19 and 8.20).

Questions arising are as follows:

(i) Can complete coincidence between poles and zeros be obtained and maintained?
(ii) If complete coincidence cannot be obtained, what are the consequences?
(iii) If the method proposed is not workable, what other approaches might be used?

Figure 8.19 An obvious strategy to cancel an unstable pole by a zero at the same location in the s plane

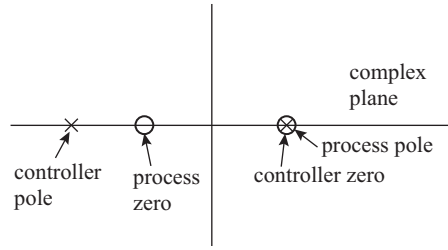


Figure 8.20 The cancellation strategy of Figure 8.19 illustrated in the complex plane

8.11.1 Cancellation of an unstable pole by a matching zero in the controller

Perfect cancellation of a pole at $s = 1$ would imply a term like $(s-1)/(s-1)$ in the overall transfer function (although it would be perfectly permissible, mathematically, to cancel both brackets, so concealing the still present potentially destabilising internal structure). However, assume that there is a mismatch of ε in the calculation so that the term above is of the form

$$\frac{s-(1+\varepsilon)}{s-1} \tag{8.17}$$

This term has the step response

$$\frac{1}{s}\frac{s-(1+\varepsilon)}{s-1} = \frac{1}{s-1} - \frac{1+\varepsilon}{s(s-1)} \tag{8.18}$$

equivalent to the time response

$$\exp(t) - \left(\frac{1+\varepsilon}{-1}\right)(1-\exp(t)) = (1+\varepsilon) + \exp(t) - (1+\varepsilon)\exp(t) \tag{8.19}$$

We see that perfect compensation implies that two exponential curves, going off to infinity in opposite directions, will precisely sum to zero (Figure 8.21).

Therefore, cancellation cannot work in practice since the instability is still present and we are relying on its effect being cancelled exactly by an equal and opposite effect.

(The differential equation would show the complete structure but the transfer function, having been subjected to cancellation, masks the true situation. This is an important point to note and shows a significant structural difference between the two different system representations that will resurface when state space representations are discussed.)

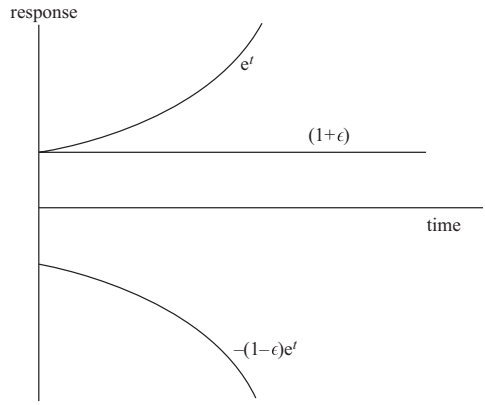


Figure 8.21 The components of the step response when there is a mismatch between pole and compensating zero

8.11.2 Shifting an unstable pole by feedback

As an alternative to attempted cancellation of an unstable pole, it may be possible to shift the pole by feedback (Figure 8.22). Taking the same unstable process as before, we examine the effect of the feedback shown. The overall transfer function is

$$\frac{s+1}{s-1+cs+c} = \frac{s+1}{(1+c)s+c-1} \tag{8.20}$$

and the system is genuinely stabilised provided that $c > 1$. The literature is fairly sparse on the control of unstable systems but see the practically oriented reference Padma Sree and Chidambaram (2005).

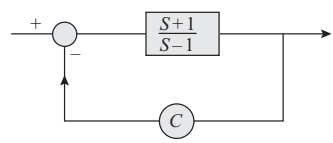


Figure 8.22 Feedback to shift an unstable pole

We can look at this same approach in another way. Consider the differential equation

$$\frac{dy}{dt} = ay + u, \quad y(0) = y_0 \tag{8.21}$$

where we imagine that a is a process constant that is outside our control and that u is under our control. Let $a = 2$ and $u = 0$.

Then, clearly, the system is unstable with an exponentially increasing response for any $y_0 \neq 0$.

Now we introduce feedback; we measure y, compare it with a desired value (here set to zero for simplicity), multiply it by a gain k and set $u = ky$, this action modifies the equation to produce

$$\frac{dy}{dt} = ay + ky, \quad y(0) = y_0 \tag{8.22}$$

or

$$\frac{dy}{dt} = (a + k)y, \quad y(0) = y_0 \tag{8.23}$$

and since the loop gain k is under our control, the system can clearly be stabilised by feedback in this way. One is bound to ask: does this strategy work in practice? The answer is a qualified 'Yes'. Qualified since one must never forget that the underlying real system remains inherently unstable and the stable behaviour is only maintained by the constant activity of the (hidden from view) control signal u and that a component failure (for example) could immediately bring back instability.

The literature is fairly sparse on the control of unstable systems, but see the practically oriented reference Padma Sree (2005).

8D Control of quiet vertical standing in humans

An inverted pendulum is the classic unstable device that can be found mounted on a motorised trolley in many control systems laboratories as a first exercise for students in designing feedback algorithms for a process that is clearly unstable. In this case, stabilisation consists essentially in rapidly compensating for every perturbation of the pendulum from the vertical by continually moving the motorised base to maintain verticality. (Many of these demonstrations make control easier to achieve by constraining the pendulum so that it can move only in one vertical plane.)

Physiologists studying the quiet standing of humans in the vertical position find the situation is quite analogous to the inverted pendulum, except that humans do not maintain verticality by rapidly shifting their feet in imitation of the agile laboratory trolley. Instead, human static balance is maintained as shown in Figure 8.23 by two main mechanisms that operate independently. See Winter (1995).

The fact that standing people who become unconscious rapidly collapse into a more horizontal position confirms the natural instability of vertical standing and also indicates that consciousness is required for that control system (summarised in Figure 8.23) to operate properly. (Although sleepwalkers seem able to successfully manage posture control on their nocturnal wanderings!)

The human body contains interrelated control and co-ordination systems allowing us not only to stand vertically but also (for example) to race across a river from stone to irregular stone or to jump so as to strike a fast-moving ball.

Within the body, generally, a remarkable hierarchy of interdependent systems from unconsciously operating cellular level up to purposeful and goal-seeking level have evolved.

When standing vertically with feet side by side, the hips compensate for lateral perturbations, whereas the ankles compensate for forward/backward perturbations

When standing vertically with feet in the tandem position, roles are now reversed, the ankles now compensate for lateral perturbations and the hips for forward/backward perturbations

Figure 8.23 Summarising how the mechanisms for maintaining static balance in the standing human change over when foot positions change

8E Understanding, with the help of the full Nyquist stability criterion (Nyquist, 1932), how simple feedback control can stabilise an unstable process

The full Nyquist stability criterion is of real value when dealing with processes that have right-half-plane singularities as in the example below.

We consider the inverted pendulum and attempt stabilisation by continually moving the base so as to prevent departure from the vertical. The ultra-simplified, linearised model is derived as follows:

A normal, non-inverted pendulum has the lossless equation

$$l\theta'' = -g\theta \tag{8.24}$$

l being pendulum length, θ being angular departure from the vertical and $g\theta$ being the gravitational restoring force. But for the inverted pendulum, we need to write

$$l\theta'' = -g\theta + u \tag{8.25}$$

where u represents the externally applied attempting balancing force.

Setting all units to values yielding the simplest possible representation, we obtain the transfer function

$$\frac{\theta}{u} = \frac{1}{s^2 - 1} = \frac{1}{(s-1)(s+1)} \tag{8.26}$$

Clearly, the open loop system is unstable, as expected, having one open loop pole in the right half plane.

We now demonstrate the extraordinary power of the Nyquist theorem by setting a proposed controller $u(s) = k(s + 4)$ so that the enhanced open loop gain becomes

$$L(s) = \frac{k(s+4)}{(s-1)(s+1)} \tag{8.27}$$

and plotting the Nyquist diagram for $L(s)$ – note carefully $L(s)$ is the open loop transfer function of controller plus process – for two different values of gain k.

From that diagram we shall obtain stability information about the expected closed loop behaviour. In principle, as in Figure 8.24, we sweep around the right half plane, following the clockwise-going contour Γ and mapping into the $L(s)$ plane to produce the two $L(s)$ contours shown in Figure 8.25.

By Nyquist's criterion, the number N, of clockwise encirclements of the minus one point in the $L(s)$ diagrams, will satisfy the relation

$$N = Z - P \tag{8.28}$$

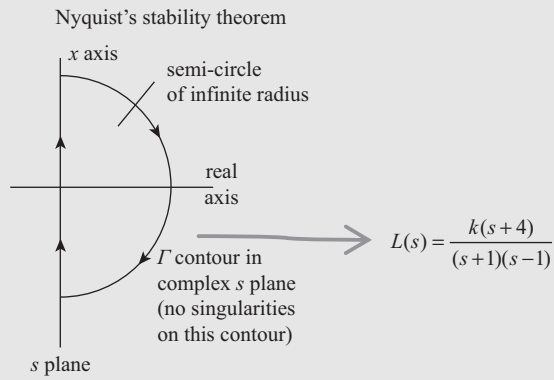


Figure 8.24 The Γ contour n in the s plane

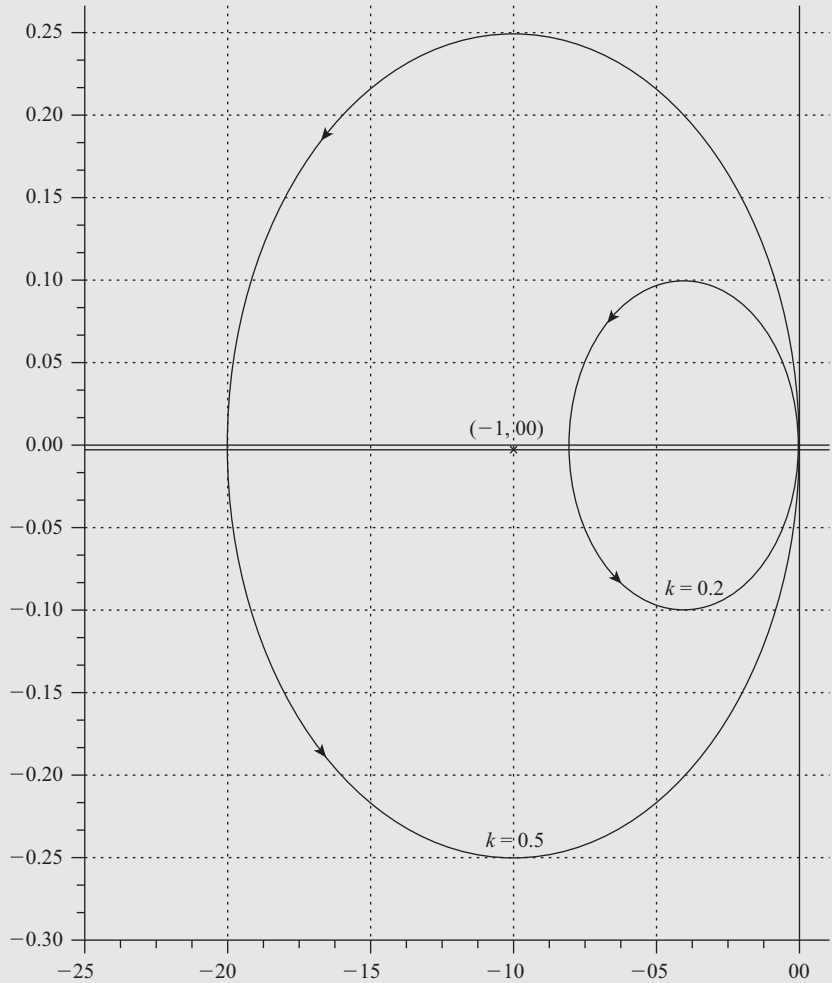


Figure 8.25 The L(s) plot in the complex plane

where Z is the number of zeros of $1 + L(s)$ and importantly therefore, also Z is the number of poles of the (proposed) closed loop system, i.e.

$$Z = \frac{L(s)}{1 + L(s)} \tag{8.29}$$

and P is the number of right-half-plane poles of $L(s)$.

If we examine Figure 8.25, we see that both $L(s)$ contours are closed ellipses with anti-clockwise orientation, but the smaller one, corresponding to a gain value $k = 0.2$, does not encircle the minus one point, therefore

yielding $N = 0$, whereas the larger ellipse, corresponding to a gain $k = 0.5$, encircles the minus one point, anti-clockwise, therefore yielding $N = -1$.

This result shows, in a very appealing topological way, that the closed loop system with $k = 0.2$ would have $Z = N + P = 0 + 1 = 1$ pole in the right half plane and therefore would be unstable.

The closed loop system with $k = 0.5$ would have $Z = N + P = -1 + 1 = 0$ poles in the right half plane and therefore would be stable. This rather beautiful result gives real insight into the nature of the problem, even with an ultra-simple model.

This type of closed loop system is unstable for low loop gains but can be stable for higher loop gains (so long as perturbations do not destroy the initial assumptions of linearity).

Now our intuition is confirmed that if you want to balance a stick, then to succeed you will surely need some minimum level of acceleration that can be equated with the minimum gain for stability that we have just demonstrated.

8F Examples of systems that are deliberately designed to be unstable

Unsurprisingly, an inherently unstable process is usually difficult to control, as earlier sections of this chapter have demonstrated, yet there are applications where an inherently unstable system is deliberately designed as the following discussion of aircraft stability illustrates.

Considering only the longitudinal stability of an aircraft, it can be shown (Abzug and Larrabee, 2002) that to achieve stability, the aircraft's centre of gravity (CG) must always be forward of the neutral point (NP), which is defined as the point in the aircraft fuselage where the orthogonal aerodynamic forces of lift and drag can be considered to act. Therefore, commercial aircraft will have their points CG and NP always disposed as shown in Figure 8.26(a). Now, although the CG is a function of the distribution of mass and for any particular aircraft is load-dependent but approximately fixed, the NP moves significantly, for aerodynamic reasons, as flight conditions change; in particular, it moves rearwards as the airspeed increases, increasing the stability margin. This means that an aircraft that is stable at low flying speeds may become excessively stable with loss of agility and manoeuvrability at supersonic speeds. Figure 8.26(b) illustrates this situation diagrammatically. Military aircraft with characteristics of instability at subsonic speeds and high agility at supersonic speeds will almost certainly

depend critically on high degrees of automated active stabilisation and guidance to assist the pilot, particularly during subsonic manoeuvres.

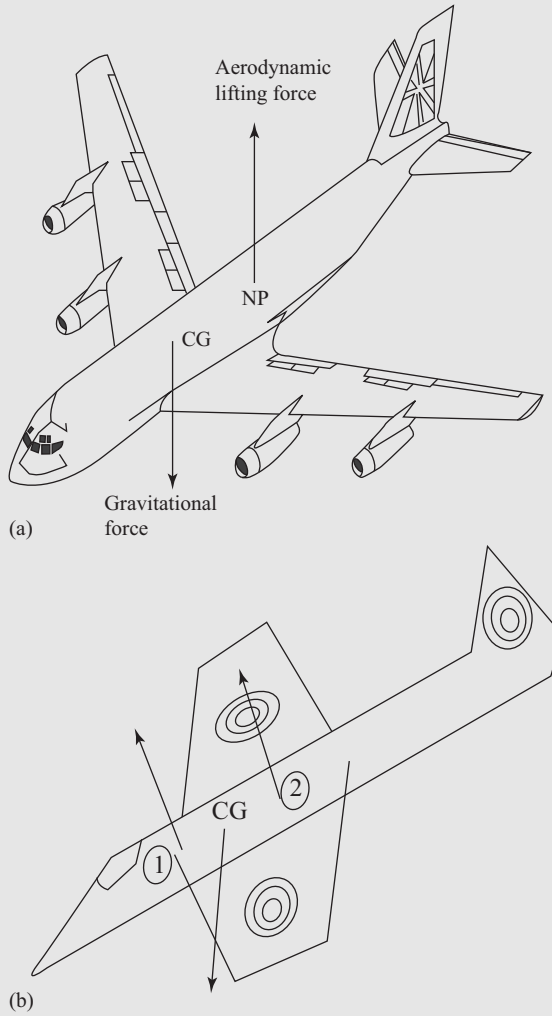


Figure 8.26 (a) The neutral point (NP) is a civil aircraft must always be behind the aircraft's centre-of-gravity (CG), to ensure longitudinal stability. (b) The NP in an advanced Military aircraft may be designed to be forward of the CG (position 1) at low speed, although that implies longitudinal instability. As, the airspeed increases, the NP naturally moves backward to position 2, ensuring stability at high speeds, but not excessive stability which implies poor manoeuvrability.

It will surprise no one that, in matters of ingenuity of design, Nature usually got there first! There are many examples of deliberately unstable systems in nature. For instance, over many centuries, fossil records confirm that flying creatures such as bats have evolved from larger stable passive 'designs', able to glide with little exercise of brain power, to smaller more efficient, but inherently unstable, agile, versions that include fast-acting closed-loop control and active stabilisation.

See 'The importance of the nervous system in the evolution of animal flight' by Maynard Smith (1952) and *The Biomechanics of Insect Flight: Form, Function, Evolution* by Robert Dudley (2002).

8.12 Systems that are difficult to control: non-minimum phase systems

Systems with this unwieldy name have the unpleasant characteristic that, when steered in one direction, they may initially respond in the opposite direction and only later move off in the required direction. For these interesting systems, we ask:

(i) What features in the mathematical model of a system lead to the behaviour described above?
(ii) What is the motivation for the 'non-minimum phase' naming of the systems?
(iii) What sort of physical phenomena are responsible for creating the non-minimum phase behaviour?

(1) Right-half-plane zeros in the system model can be identified with the behaviour (or for a discrete time model, Z plane zeros outside the unit circle).

Example The model

$$10_{y_k} = 9_{y_{k-1}} - u_{k-1} + 2u_{k-2} \tag{8.30}$$

has the pole–zero diagram shown in Figure 8.27(a) and the step response of Figure 8.27(b).

(2) Systems having no right-half-plane singularities are called minimum phase systems. Systems having right-half-plane singularities are called non-minimum phase systems. Therefore, we say that a strictly stable system is minimum phase if it has no finite zeros in the right half plane.

Caution: Clearly the numerators $(1 + s)$ and $(s + 1)$ are identical. However, the numerators $(1 - s)$ and $(s - 1)$ are very different in their phase characteristics. The first goes from 0° to -90° with increasing frequency, whereas the second goes from $+180^\circ$ to $+90^\circ$ with increasing frequency.

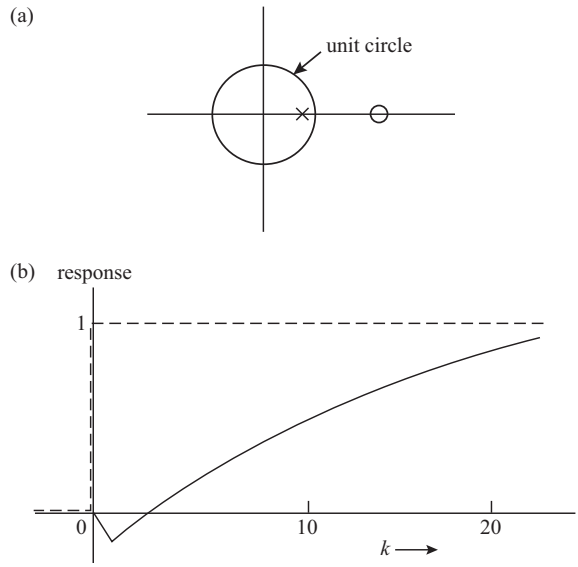


Figure 8.27 (a) Pole–zero diagram for a simple non-minimum phase system; (b) step response of the system whose pole–zero diagram is in (a)

(3) Physical phenomena that give rise to non-minimum phase behaviour. It is usually possible to correlate non-minimum phase indicators in mathematical models with physical phenomena. Examples:
- Control of the level of a volume of boiling water. When cold water is added to raise the level of a mass of boiling water, the initial effect is the collapse of bubbles with consequent initial fall in water level.
- Hydroelectricity generation. A requirement to increase the level of generated power from certain hydroelectric configurations results in an initial decrease in power during the time that the water in the pipeline feeding the turbines accelerates to the necessary increased velocity.
- Sequences of interacting processes. Suppose that a sequence of interacting processes is operating in a steady state and that it is to be brought to a new steady state. Quite frequently the transient behaviour will move in the opposite direction to that intended. In a general sense this is because, at a call to increase activity, early processes in a chain immediately use additional shared resources, whereas the benefits of their increased activity take time to work through the system.
- Spatially distributed systems, being limiting cases of interconnected processes, often exhibit non-minimum phase characteristics.

8.12.1 The effect of system zeros on the step response of a system

System zeros affect the dynamic behaviour of the system, usually adversely, compared with that of the equivalent system without zeros. Step responses either

have characteristic 'inverse responses', meaning that, initially they move off in the opposite direction to what is required or that, despite all the poles being real, the step responses exhibit overshoot.

These can be serious problems that are difficult to overcome algorithmically and, as with dead-time problems, they should be engineered out of the system at the design stage if at all possible.

Ships have many interesting problems of motion control and stabilisation. See, e.g. Chapter 8 of Perez (2005) and Hearns and Blanke (1998) (Professor Mogens Blanke having kindly supplied Figure 8.28).

Many other large-scale processes such as hydroelectric power stations, with their huge masses of moving water, have their right-half-plane zero characteristics, caused because a request for more power involves increased opening of a sluice gate and an initial fall in power during the time that the additional water is accelerated to the new required velocity.

A demonstration of the effect of numerator zeros (right and left plane) on dynamic behaviour.

The step response of the standard second-order system

$$G_n(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{8.31}$$

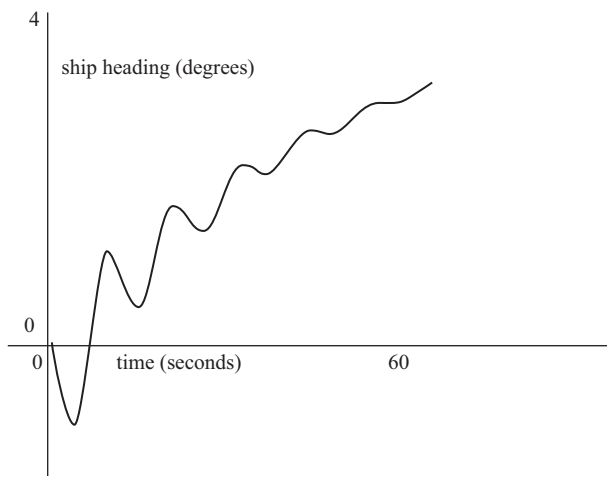


Figure 8.28 *The simulated response of a large ship when making a turn of a few degrees. Right-half-plane zeros in the ship's dynamics cause the (presumably) disconcerting inverse response of a degree or so. The oscillation is due to interaction between different dynamic modes*

with behavioural rather than physically meaningful parameters has been discussed and plotted for different values of damping factor ζ in Section 5.13.

Here we shall discuss systematically the behaviour of the same system enhanced by a zero. The ingenious approach used here is due to Åström (2008). The enhanced system is

$$G(s) = \frac{\omega_n}{\beta} \frac{s + \beta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{8.32}$$

where β is a numerical parameter that will be used to make clear the effect of the location of the system zero on the step response.

We can write

$$G(s) = G_n(s) + \frac{1}{\beta\omega_n} sG_n(s) \tag{8.33}$$

with the left term $G_0(s)$ being the familiar (no zeros) second-order system and the right term being responsible for the additional response terms caused by the zero.

We can also write

$$h(t) = h_n(t) + \frac{1}{\beta\omega_n} \frac{dh_n(t)}{dt} \tag{8.34}$$

Two things are clear:

(1) In the time domain, the right term represents the derivative of the left term (multiplied by two constants). What it means (Åström is always, admirably, searching for structural meaning amongst the numbers) is that wherever the derivative of the step response of the no zero system $h_n(t)$ is zero, the right-hand term in the above equation will also be zero. This means that the whole family of step response curves for different β must necessarily coincide at such a point, as we shall attempt to demonstrate.
(2) Reducing the values of β can be expected to increase the effect of the added zero.

Below is the step response of the second-order system whose transfer function is

$$G(s) = \frac{\omega_n}{\beta} \frac{s + \beta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{8.35}$$

for the case with no zero present and for two values of β. It is clear that negative values of β result in a so-called inverse response, whereas positive values of β cause overshoot (Figure 8.29).

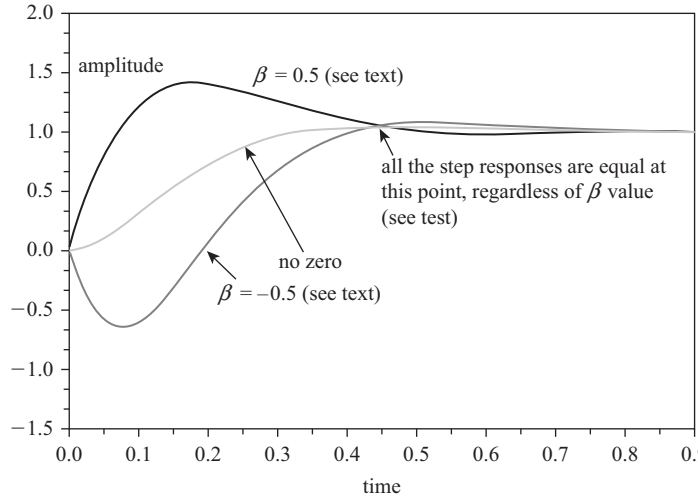


Figure 8.29 The effect of a zero on the step response of a linear second-order system

General points: For a minimum phase system, the two components of the frequency response (i.e. gain and phase) are related by a known fixed bijective function – effectively meaning that either of the components contains all the frequency response information that exists. This fact is exploited in Bode's theorems on stability (see Chapter 5).

8.13 Process dead time: a difficult dynamic element in the control loop

Dead time (also known variously as pure delay; time delay, transport lag; distance–velocity lag) is the name given to a process element whose output in response to an input is simply the input shifted by the time delay T_d.

(Clearly, industrial processes are rarely so obliging as to remain constant for long, and in practice, process dead time T_d will usually vary, sometime quite widely, complicating the control systems designer's task. However, in the discussion that follows, T_d will be assumed to stay constant.)

An incoming periodic signal of frequency ω will, on encountering a dead-time element, be phase shifted by the angle ωT_d and it is clear that for some frequency ω that phase shift will be exactly π radians. Assuming that the dead-time element was the only dynamic element to be considered, this would obviously mean continuous oscillation would occur if feedback control with unity gain was attempted. Figure 8.30 helps to explain the above discussion.

In physically large processes involving flows of liquids or material, significant dead time frequently appears because of the time taken for information to reach measuring sensors or for mixing or some other physical or chemical process to occur.

Although dead time may not feature prominently in control text books, it is nevertheless, from a control point of view, a dominating feature of many real-world industrial problems.

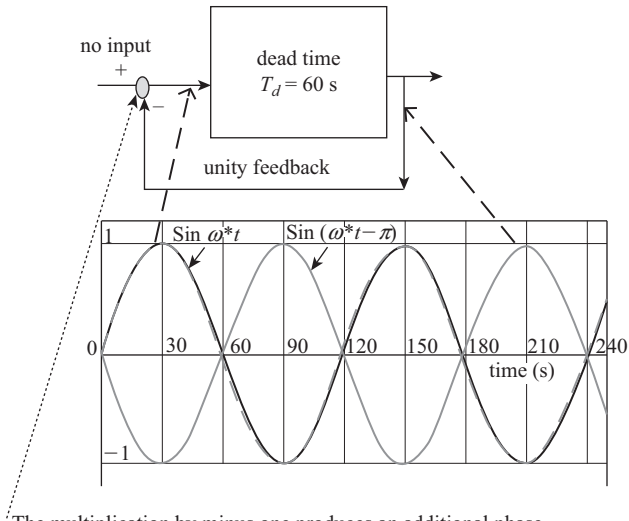


Figure 8.30 A process dead time of $T_d = 60$ s in a unity feedback loop leads to continuous oscillation with a period of $2T_d$

The presence of significant dead time in a process will severely limit what even the best-designed control can achieve and ideally a 'control presence' at the process design stage should ensure that all possible steps are taken to 'design out' all but the most unavoidable sources of dead time.

Note: The transfer function of a dead-time element is e^{-sT}, which, being irrational, may not be acceptable by some software programs. In that case, the Padé first-order approximation

$$e^{-sT_d} \approx \frac{1 - sT_d}{1 + sT_d} \tag{8.36}$$

is often substituted.

(Notice that this approximation to dead time has a right-half-plane zero indicating a family resemblance between the two difficult process elements, dead time and minimum phase elements.)

8.13.1 How to control processes that have significant dead time

The text book approach is to use the old established Smith predictor or a more recent algorithm based on the same principle that is basically to use a process model (which may be learned or corrected online via a loop that makes use of the delayed measurement) in real time to estimate the undelayed process output and to use that to drive a feedback loop. Figure 8.31 shows the principle.

Clearly, this is a sound methodology under ideal conditions but rather than blindly applying this or any other algorithm, it is usually preferable to look in detail

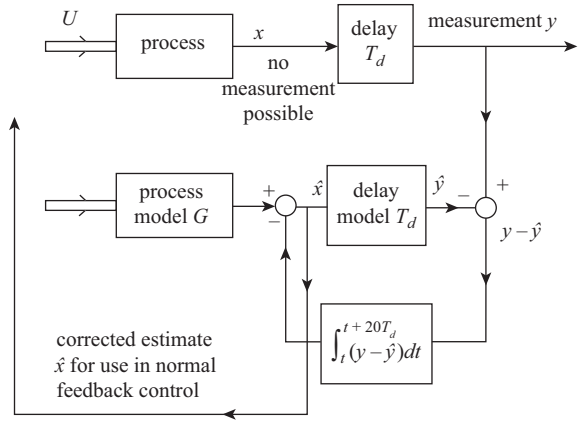


Figure 8.31 The general principle for control of a process with a significant dead time caused by a measurement delay

into the fundamentals of the process and to develop, just for this purpose, a process-knowledge-based model that can generate an alternative near-instantaneous pseudo-measurement to be used in the feedback loop, leaving the actual, delayed, measurement in the role of model calibrator.

Each application tends to be so specialised that in-depth process knowledge is much more important than a knowledge of or expertise in control theory. Although every application is unique with widely differing features, there is nevertheless a generic pattern to all. Therefore, one particular application where dead time has been successfully overcome will be described in detail as a case history in Appendix A towards the end of the book.

8.14 Some interesting theoretical limitations on performance

It is well known that Shannon's theorem sets a fundamental upper limit on the maximum error-free capacity of a communication channel. Less well known but important in the control field are a number of other fundamental design limitations, of which examples will now be given.

8.14.1 Sensitivity functions and their interrelation

(These interrelations play a major role in the loop-shaping techniques that will be introduced in Chapter 15.)

8G Motivation for the name: non-minimum phase systems

Consider first the 'usual' system of transfer function

$$G_1(s) = \frac{(1+sT_1)}{(1+sT_2)(1+sT_3)} \tag{8.37}$$

and compare it with the transfer function

$$G_2(s) = \frac{(1 - sT_1)}{(1 + sT_2)(1 + sT_3)} \tag{8.38}$$

It is clear that both transfer functions yield identical plots of magnitude as frequency varies.

However, the phase plots differ markedly, for, as the phase plot corresponding to the $(1 + sT_1)$ term in G_1 moves from 0° to $+90^\circ$ so the phase plot for the $(1 - sT_2)$ term in G_2 moves from 0° to -90°. Thus, the high-frequency asymptote for the phase angle is -90° for G_1 but -270° for G_2.

Alternatively, consider

$$G_3(s) = \frac{(s - 2)}{(s + 2)} \tag{8.39}$$

This has constant magnitude at all frequencies but the phase angle is $+180^\circ$ at low frequencies, decreasing to 0° at high frequencies.

If two transfer functions are strictly stable with the same gain at each frequency, then the one with all zeros in the left half plane will have least phase shift. Figure 8.32 illustrates the point.

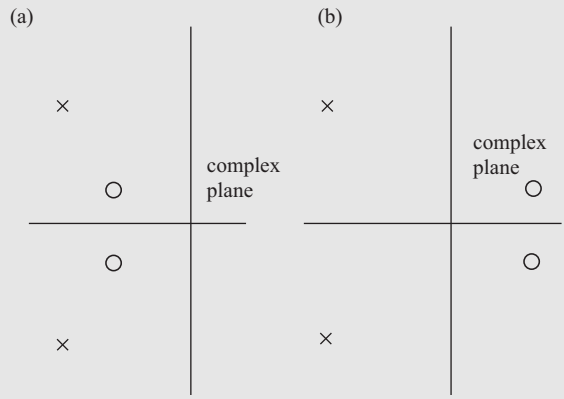


Figure 8.32 (a) The pole–zero diagram for a normal (minimum phase) system; (b) the pole–zero diagram for a non-minimum phase system that has the same characteristics as the system in (a)

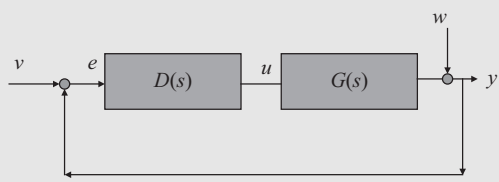


Figure 8.33 Feedback configuration

Consider a process $G(s)$ in a closed loop with a controller $D(s)$ (see Figure 8.33).

We define two dimensionless sensitivity functions T and S as follows

$$S = \frac{1}{1 + SD} \quad T = \frac{GD}{1 + GD} \tag{8.40}$$

and note that at any frequency ω where $T(\omega) = 1$, we will have $y = v$, i.e. output = desired value.

Thus, T links output y with desired value v, whereas the function S links disturbance output y with the disturbance input w.

Relations between T and S and their consequences:

By inspection,

$$S(s) + T(s) = 1 \text{ for all } s \tag{8.41}$$

This relation can be regarded as a constraint on design, preventing independent choices being made in regard to reference following and disturbance rejection performances.

8.14.2 Integral constraints in the time domain

Example 1 If the open loop combination $G(s)$, $D(s)$ has the form

$$\frac{P(S)}{s^2 Q(s)} \tag{8.42}$$

i.e. has two poles (a double integrator) at the origin, assume the closed loop to be stable. Then, irrespective of what other (linear) elements the brackets in (8.42) contain, the error $e(t)$ following the application of a unit step applied at $t = 0$ must satisfy the relation

$$\int_0^\infty e(t)dt = 0 \tag{8.43}$$

so that equal areas of positive and negative error must result as indicated in Figure 8.34.

Illustration of the effect discussed as Example 1

Assume that

$$GD = \frac{10s + 16}{s^2} \tag{8.44}$$

so that

$$\frac{GD}{1 + GD} = \frac{10s + 16}{s^2 + 10s + 16} \tag{8.45}$$

with poles at -2, -8 and a step response in the time domain as shown in Figure 8.34.

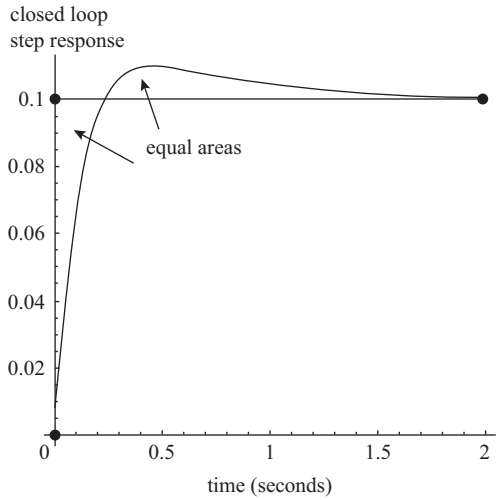


Figure 8.34 The closed loop step response of the open loop system $G(s)D(s) = (10s + 16)/s^2$. *Note the equal areas marked, confirming that the double integrator leads to the error e(t) satisfying the equation* $\int_0^\infty e(t)dt = 0$, *e(t) following the application of a step at* $t = 0$

If the open loop combination GD has right-half-plane poles or zeros, then evaluation of the integral

$$\int_0^\infty e(t)dt \tag{8.46}$$

following the application of a step will, in each case, show that there are inevitable under- and overshoots in the closed loop responses, so that for instance, when a real open loop zero is present in the right half plane then the step response will inevitably begin with a negative-going response that is typical of so-called non-minimum phase systems (see interlude 8G).

8.14.3 Design constraints caused by Bode's theorem

Bode's theorem states that

$$\int_0^\infty \ln|S(j\omega)|d\omega = 0 \tag{8.47}$$

This shows that the average value of the sensitivity function S must be 1 on the imaginary axis so that if very small values of S are forced on the system for some range of frequencies, values greater than 1 will have to be accepted as payback over some other frequency range.

If one imagines that the loop can be shaped so that the undesirably high values of S occur at frequencies well outside the system bandwidth, this strategy turns out to be prevented by other constraints as Seron *et al.* (1997) shows (this is yet another manifestation of the well-known NFL (no free lunch) syndrome!).

This section is based on Seron *et al.* (1997), an interesting and comprehensive reference where more results can be found, and on Freudenberg and Looze (1985). Bode's theorem can be found in Bode (1945).

8H Mapping of complex functions: a few points that underlie classical control theory

Given $y = f(x)$, where x and y are real scalar-valued functions, there is only one path for x to follow, i.e. from $-\infty$ to ∞ and the resulting value of y is the usual 'graph' of y against x. No variation is possible.

However, for a complex (valued) function, $g = f(s)$, with complex argument s, the values taken by g depend on the path chosen for s in the complex plane. For instance (McCollum and Brown, 1965, p. 85), if s is allowed to vary as shown in Figure 8.35(a), then $G(s) = 10/(s - 2)$ varies as shown in Figure 8.35(b).

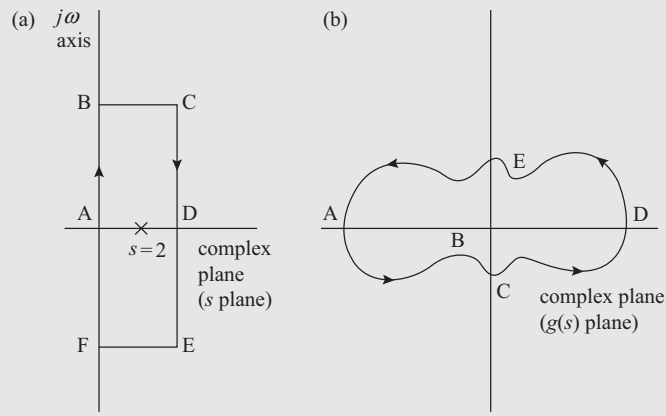


Figure 8.35 (a) A path in the complex plane; (b) the corresponding path for $G(s) = 10/(s - 2)$

Notice that the left contour encircles the pole at $s = 2$ in a clockwise direction, whereas the corresponding contour for g encircles the origin of the complex plane in an anti-clockwise direction. Further investigation would show that the direction of rotation of the g curve and its encirclement (or not) of the origin is directly related to the presence or absence of poles and zeros within the region that is encircled by the s curve. Figure 8.36 gives further examples.

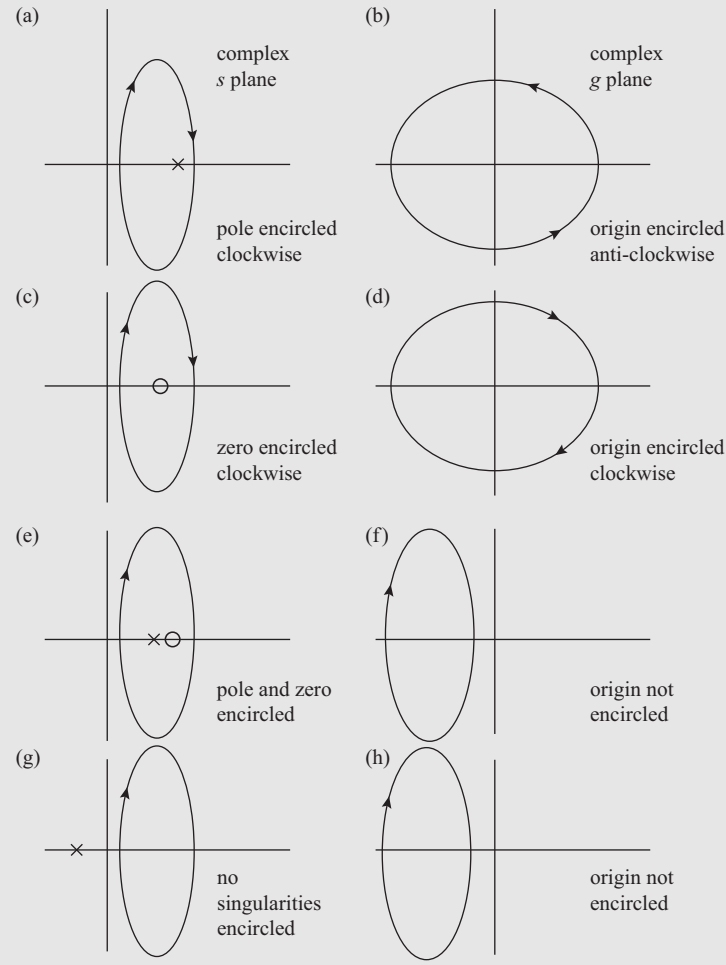


Figure 8.36 The left-hand diagrams (a), (c), (e), (g) show paths in the complex s plane. The right-hand diagrams (b), (d), (f), (h) show corresponding paths in the G(s) plane

The foregoing material is part of the subject 'functions of a complex variable', which underpins all of the control work (stability, poles and zeros, etc.) that relies on transfer functions.

Returning to the mapping and encirclement discussion, if s is allowed to encircle the right half of the complex plane, then the behaviour of the transfer function $G(s)$, as s varies, can indicate the presence of poles in that region. Since such poles imply system instability, this idea forms the basis for a major stability test – the Nyquist criterion.

Because we are interested principally in negative feedback systems, the function that we need to consider is not really $G(s)$ but rather $G(s)/[1 + G(s)]$. The form of the denominator shifts the emphasis from the origin to the point $-1 + j0$; this is the point whose encirclement or non-encirclement yields stability information for feedback systems.

8I Derivatives of a complex function $G(s)$

Not all complex functions are well behaved in the complex plane. Some are able to possess more than one value of derivative at the same point, according to the direction in which s is varied. Such behaviour is not possible when the function satisfies the Cauchy–Riemann conditions at almost all points in the plane. The function is then called an analytic function.

8J Singularities of a complex function $G(s)$

Singularities are the points at which G, or its derivatives, do not exist. The location and nature of the singularities determine the behaviour of the function in the entire plane.

There are three types of singularities: poles, essential singularities and branch points. If a positive integer n exists such that

$$\lim_{s \to s_1} (s - s_1)^n G(s) = k \tag{8.48}$$

Where k is some finite non-zero value, then s_1 is a pole of $G(s)$ of order n.

An essential singularity, roughly, is a pole of infinite order. In control theory, essential singularities usually arise as models of dead-time processes.

A branch point is associated with a multi-valued function such as $\sqrt{s}$.

Behaviour of G(s) near to a pole:
$G(s)$ may be expanded in a Taylor series about a pole at s_1 as

$$(s - s_1)^n G(s) = A_{-n} + A_{-n+1}(s - s_1) + \cdots + A_{-1}(s - s_1)^{n-1} + B_0(s - s_1)^n + B_{01}(s - s_1)^{n+1} + \cdots \tag{8.49}$$

Hence,

$$G(s) = \frac{A_{-n}}{(s-s_1)^n} + \frac{A_{-(n-1)}}{(s-s_1)^{n-1}} + \cdots + \frac{A_{-2}}{(s-s_1)^2} + \frac{A_{-1}}{(s-s_1)} + B_0 + B_1(s-s_1) + \cdots \tag{8.50}$$

which is called a Laurent series (study of the Laurent series and its connection with the behaviour of functions in the time domain can be pursued in Truxal (1955), pp. 4–29).

A_{-1} is called the residue of $G(s)$ at s. Near to the pole, the term in A_{-1} dominates the series.

Note: Source material and suggestions for further reading to support the topics of this chapter will be found in Chapter 18.

Chapter 9

Some practical aspects of control design, implementation and justification

9.1 The variables that really matter may be difficult to define, measure or control

In real-world situations it is rare for the variables that are most important for success/viability/profitability to be agreed upon, well defined and measurable.

Taking an everyday example, in the care of an individual patient by a medical doctor, what matters most could perhaps be defined as the patient's well-being. That complex variable is obviously not easily inferred from the readily available measurements of pulse rate, temperature, heart rate and brain activity. In practice, information on well-being is often provided by patient–doctor dialogue, while the success or otherwise of a treatment by drugs may again be assessed mainly by dialogue.

Industrial situations are not so different. In a typical industrial process, the online available measurements will usually be rather peripheral to the variables that are needed for control or that relate to 'customer satisfaction' (in the widest sense). For example, it is all too easy to measure the (unimportant) temperature of the flames inside a heating furnace but all too difficult to measure or even estimate, as a function of time, the internal temperature contours of a massive object that is being heated. Important (to the customer) product variables, (say) the clarity of plate, glass or the texture and taste of a food product, are typically difficult to define and even more difficult to measure online.

Further, the effects of those many interacting, constrained, manipulable and non-manipulable, known and unknown process variables that influence product quality are not easily quantified or modelled.

9.2 How efficient is it to control a process by a controller that consists only of a high gain of value C that, in the limit, becomes a relay controller?

9.2.1 Rudimentary on–off control

In view of the evident efficiency of feedback controllers in controlling unknown phenomena, is it not feasible to attempt control of all processes by some very simple standard strategy?

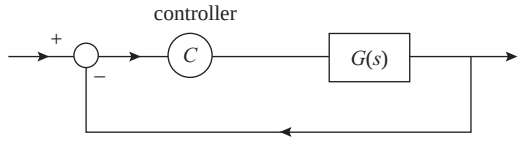


Figure 9.1 The simplest possible controller – a gain C

The simplest possible controller (Figure 9.1) involves just multiplication of the error by a scalar C; the overall transfer function is $CG(s)/(1 + CG(s))$ and if C is very high, then the overall transfer function is approximately

$$\frac{CG\,(s)}{CG\,(s)} = 1$$

i.e. provided that $C \gg 1$, near-perfect control can be obtained.

Question: What happens as $C \to \infty$? Will this give better and better control?

Answer:

(i) As C is increased, the system *may* become unstable and unusable.
(ii) Assuming that the system remains stable as $C \to \infty$ (another question left for the moment is when does this arise?), then we have arrived at a switched (relay) control system (Figure 9.2). Such a system does indeed have a high performance, and the low cost of a switching controller also makes such systems economically attractive.

However, there are two disadvantages of (infinite gain) switching systems:

(i) They are essentially non-linear (for instance, they respond (initially) in the same way to the input step $v = 1$ as to the input step $v = 10$).
(ii) The system never, under any circumstances, comes to rest: full power, in one direction or the other, is always being applied. For many applications, such behaviour is not acceptable.

Summary: A controller that consists only of a high gain C may give good control of a totally unknown process, though the upper bound for C may be set at a low value by stability considerations.

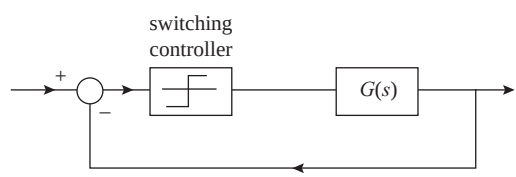


Figure 9.2 The limiting condition: as $C \to \infty$, the controller becomes a relay

Where stability conditions allow, increasing the gain C will eventually result in a relay as the controller. Such a relay does indeed frequently give good control of an unknown process but brings problems (non-linearity, continuous oscillation) of its own.

Despite these disadvantages, relay control, also known as on–off control, has significant practical advantages that lead to its being widely applied across industry. The chief of these advantages is the very low cost of on–off actuators, compared with the continuously variable actuators needed for continuous control. On–off control manages to be surprisingly versatile; for instance, it can

(i) achieve temperature control of a gas-fired furnace by switching between high and low gas/air flow rates using only a pair of simple solenoid valves;
(ii) operate conveyors or other large material handling devices at any chosen average flow rate by alternately switching between two different ratios of a gearbox;
(iii) achieve continuously variable control of many devices, such as electric motors, by on–off modulation of an electrical power supply. For large applications, the savings achieved by avoiding the need for continuously variable amplifiers/actuators often outweigh any disadvantage of the discontinuous operation.

Rapid advances in continuously variable actuators have considerably reduced, but not eliminated, the cost advantages of on–off (relay) systems, particularly for physically large systems.

Relay control systems can be analysed and designed using phase plane and describing function methods – see Chapter 7 – and there is a specialist methodology for relay control systems that can be found in, for instance, Kochenburger (1950) and Flugge-Lotz (1953), two of the pioneers in the field. Tsien (1954) devotes an interesting chapter to the topic as do many of the older books on non-linear control.

9.2.2 *Introduction to variable structure systems and sliding mode control*

In Section 9.2.1, we discussed the use of simple on–off control via a straightforward feedback loop. Here we demonstrate how, with the application of switching logic, interesting further possibilities can be exploited. We shall introduce switching lines, switching surfaces, sliding modes and variable structure systems.

Whereas theoretical aspects of simple relay control are seen largely as only of historical interest, sliding mode control is a current research topic.

Consider a feedback loop where the controller is a simple switch whose only output is $+C$ or $-C$, where C is a constant gain. The switch-over from $+C$ to $-C$ will usually be designed to depend on some function of the error in the loop. The line in the phase plane, where logic changes the sign of C in such a controller, is called a **switching line,** or in higher dimensions, a **switching surface.**

The following illustrative examples are reproduced by kind permission of Professor Stanislaw Zak from the reference Zak (2003):

Consider the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = C$$

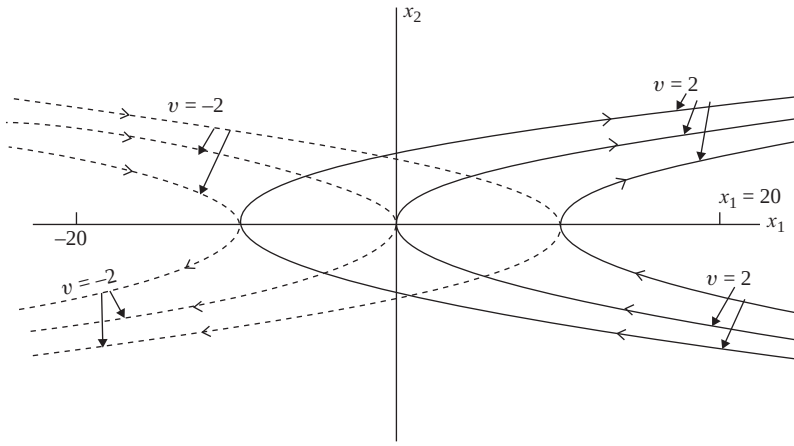


Figure 9.3 Phase-plane diagram showing the two families of solutions for the two possible choices $C = 2$, $C = -2$

where C is constrained to take on only the values ± 2.

The following phase plane sketch shows the two families of solution curves yielded for selected initial conditions for the two possible choices $C = 2$, $C = -2$ (Figure 9.3).

It is clear that both sets of solutions are unstable with solutions that, because of the double integration, include ever increasing terms in both t and t^2.

However, once a switching line is introduced that switches the sign of u, all trajectories that encounter it, at least in the neighbourhood of the origin, are captured onto it, after which they slide towards the origin in what is called **sliding motion** (Figure 9.4).

But what is really happening as the system state progresses along the switching line in such a satisfactory manner and what is the value of C during that progress?

From an idealised point of view, the value of C is moving between its two allowable values infinitely often and the state stays exactly on the line as it progresses towards the origin.

In all practical cases, no such idealised switching is possible and looking in fine detail at behaviour on the switching line soon reveals that the system is following a discontinuous trajectory made up of very many tiny elements of $C = 2$ trajectory, alternating with $C = -2$ trajectory as shown in Figure 9.5.

From two unstable systems of little practical interest, with the aid of simple logic and some knowledge of sliding mode theory, we have constructed a simple but potentially valuable control system that would be expected to be robust, since factors such as model uncertainty presumably cannot influence the behaviour significantly.

Our next illustration, again provided by Professor Zak, makes the point that two systems that are only marginally stable can, when controlled by sliding mode logic,

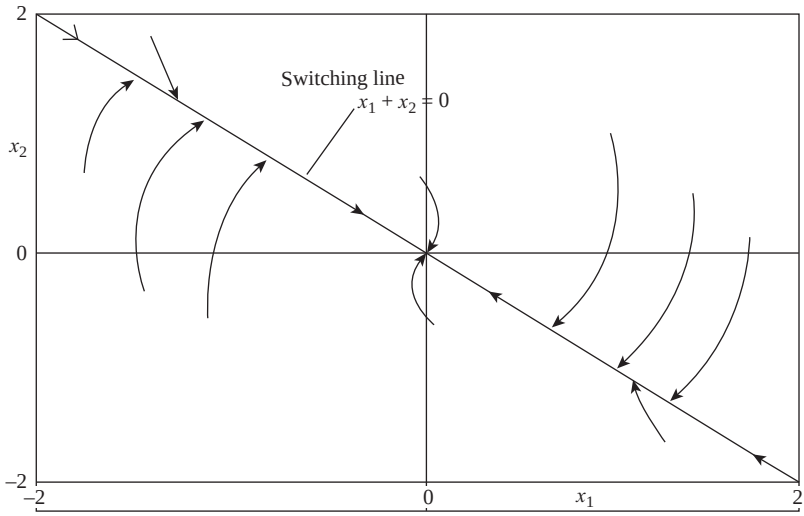


Figure 9.4 All possible trajectories in a region around the origin will be captured onto the switching line and then will approach the origin by sliding motion

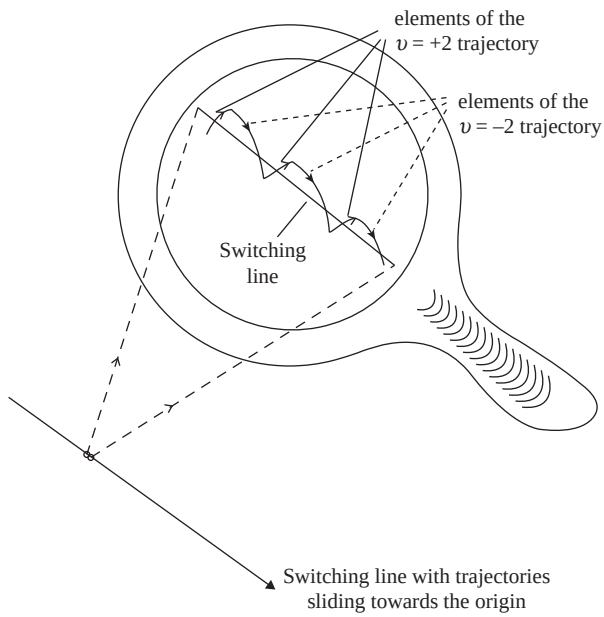


Figure 9.5 A magnified view, showing that when a trajectory theoretically slides along a surface, it is, in any actual realisation, not sliding, rather it is switching rapidly between alternate trajectories at a frequency determined by the magnitude of the delays (unmodelled in our treatment) in the particular physical realisation

combine to produce a third, stable system. Such possibilities are the reason for the name **variable structure control** that is sometimes attached to sliding mode approaches.

This example (which Professor Zak actually credits to Utkin (1977) as the original source) is as follows:

Consider the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -ux_1$$

where u can take only two possible values, $u = 1/5$ for case (a) or $u = 5$ for case (b).

The two cases are sketched in the phase plane in Figure 9.5 where it can be seen that both systems have trajectories consisting of families of ellipses; certainly not asymptotically stable to the origin in either case.

Next, switching logic is introduced, such that

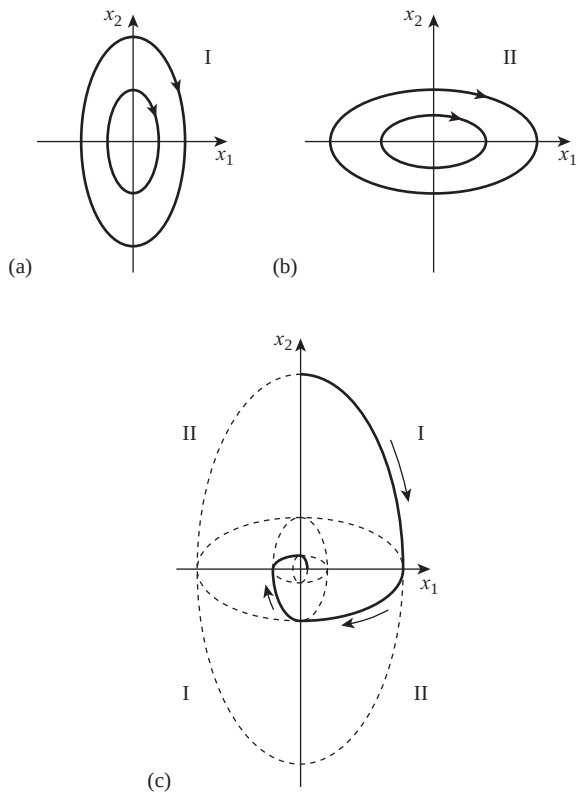


Figure 9.6 Phase plane sketches for (a) the system with u = *1/5; (b) the system with* u = *5; (c) the system with* u *controlled by switching logic depending on the sign of* x_1x_2

u takes the value 1/5 if $x_1x_2 < 0$

u takes the value 5 if $x_1x_2 > 0$

With the introduction of this switching logic, a new stable structure is produced, as shown in Figure 9.6(c).

A significant current research literature exists often aimed at practical exploitation of sliding mode control in applications where significant in-process variations occur. A typical paper is 'Real-time application of discrete second order sliding mode control to a chemical reactor' by Milhoub (2009). See also the book Misawa (2012).

9.3 An off-the-shelf approach to the control of an unknown process

Perhaps 90% of control problems encountered in industry can be solved routinely and do not require an extensive modelling and control design exercise. For such processes, a fixed-structure, commercially purchased, ***three-termcontroller*** will probably prove adequate. Such devices can be discrete instruments fixed in racks or they may be invisible library algorithms within an overall monitoring and control package.

9.3.1 The three-term controller

(Three-term controllers are very frequently referred to simply as PID controllers: PID being an abbreviation for Proportional, Integral and Derivative; describing the actions performed by the three terms that go to make up such a controller.)

Three-term controllers are the control practitioners' everyday workhorses. They are highly successful in practical situations but are looked down upon by theoreticians and are not even mentioned in many undergraduate texts. The idea of a three-term controller, already introduced in Section 5.10, is as follows:

(i) To use a gain C that is to be set not too high, to avoid the problems of non-linearity and continuous oscillation that can arise from too high a C value.

(ii) To add an integrator into the controller to ensure that, regardless of the value of C, a constant desired value v will result (after transients have died away) in a constant measured value y, with y being exactly equal to v.

(iii) To add a differentiator into the controller to give independent control of the degree of damping. If this appears to be a rather unadventurous low technology solution, it is worth remembering two things:

   (1) Non-linear controllers are quite a rarity; hard experience having shown their limitations. If one is restricted to using a linear controller, then a combination of multiplication by a constant, differentiation and integration pretty much spans the range of what one can undertake and still remains linear!

(2) A very high proportion of essential control loops across a wide span of industry is under efficient and reliable PID control, a fact confirmed in the paper Desborough and Miller (2002), which reported that 97% of more than 10,000 process controllers surveyed were under PID control.

9.3.2 Illustration of the value of an integral term in removing any constant error

Assume that the process to be controlled has the transfer function

$$G(s) = \frac{1}{s+1}$$

In closed loop in series with simple controller of gain C, the steady state response to a unit step as $t \to \infty$ is

$$\frac{C}{s+1+C} \text{ as } s \to 0 = \frac{\mathrm{C}}{1+C}$$

Thus, for finite C, there is a constant error of $1/(1 + C)$. When an integrator is added to the controller (in parallel with the gain C), the steady state response to a unit step is

$$\frac{sC+1}{s(s+1)+sC+1} \to 1 \text{ as } s \to 0$$

i.e. with the integrator present, the steady state error is zero.

9.3.3 Illustration of the value of a derivative term to control the degree of damping

The transfer function of the closed loop system of Figure 9.7(a) is

$$\frac{C}{(s+1)(s+3)+C}$$

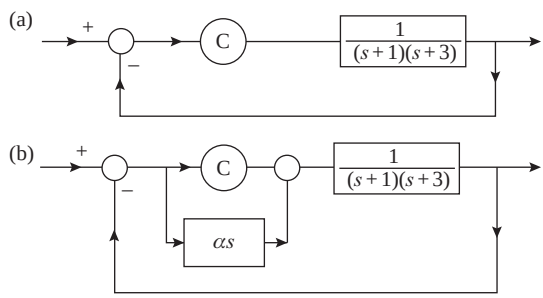


Figure 9.7(a) A system under closed loop control with a simple controller of gain C; (b) the system of (a), enhanced by a derivative term

If we now fix C at some numerical value, say $C = 65$, the closed loop poles will be located at

$$s = -2 \pm \sqrt{1 - C} = -2 \pm j8$$

Very light damping is indicated by these pole positions.

If, now, referring to Figure 9.7(b), a derivative term αs is included in the controller, then the closed loop transfer function becomes

$$\frac{C + \alpha s}{(s+1)(s+3) + C + \alpha s}$$

and, keeping the value of C set at $C = 65$, it is found that the closed loop poles are now located at

$$s = -\left(2 + \frac{\alpha}{2}\right) \pm \sqrt{\left(2 + \frac{\alpha}{2}\right)^2 - 3 - C}$$

And it can be seen that, by choice of α, the poles can be moved to positions giving any required degree of damping, although of course the effects of the introduced zero on overall performance will need to be considered.

This ingenious diagram (Figure 9.8), which is reproduced here by the kind permission of Professors Åström and Hägglund (2006), shows how at any time t the output of a PID controller is made up of

- the **proportional term** equal to the **instantaneous** error;
- the **integral term** equal to the **summation of all the past error** (that is indicated by the shaded area in the figure);
- the **derivative term** equal to an **estimate of future error, based on a linear prediction**,obtained by projecting the instantaneous slope forward for a time T_d.

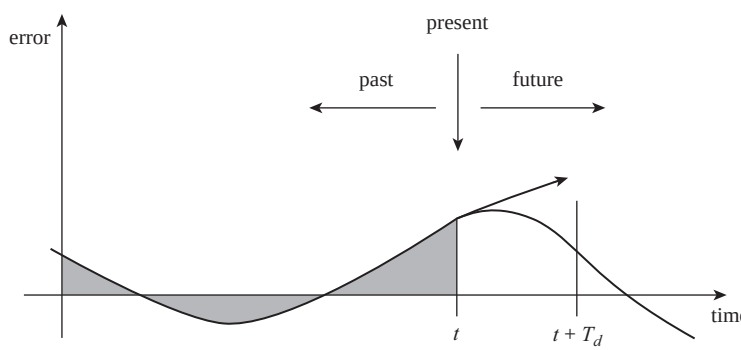


Figure 9.8 Visualisation of the principle of action of a three term controller

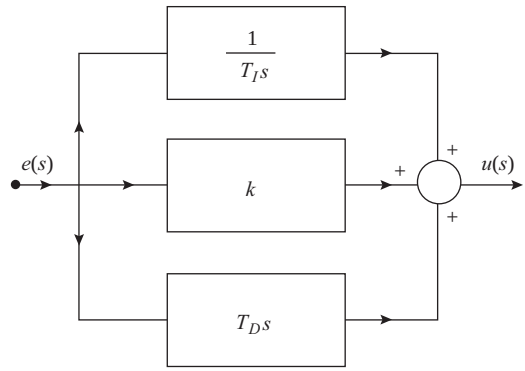


Figure 9.9 A three-term controller

9.3.4 How can the three coefficients of a three-term controller be chosen quickly in practice?

For most processes that need to be controlled, we cannot expect to have available an accurate or even an approximate model, since modelling is an expensive and time-consuming procedure.

For routine situations, all we wish to know is how to set the three coefficients: gain, derivative action, integral action, that are required by the three-term controller (Figure 9.9). There are three basic approaches.

9.3.4.1 To apply a step to the process that is to be controlled and use the response to calculate the coefficients

We shall outline that approach and give an illustrative example.

This approach is simple and reliable but it does require that the process is available and at one's disposal to have an open loop step test performed. The procedure is as follows. The process, regardless of its actual (and in any case usually unknown) structure will be modelled by the approximation

$$G'(s) = \frac{Ke^{-sT_2}}{(1 + sT_1)} \tag{9.1}$$

i.e. by a first-order system in series with a finite time delay T_2. The three coefficients K, T_1, T_2 are read off from the open loop step response of the process using the graphical construction shown in Figure 9.10.

The three controller coefficients are then found from the Ziegler–Nichols (1942) equations

$$\begin{aligned} &\text{Controller gain } C = \frac{1.2}{K}\frac{T_1}{T_2} \\ &\text{Integral time constant } T_I = 2T_2/C \\ &\text{Derivative time constant } T_D = 0.5CT_2 \end{aligned} \tag{9.2}$$

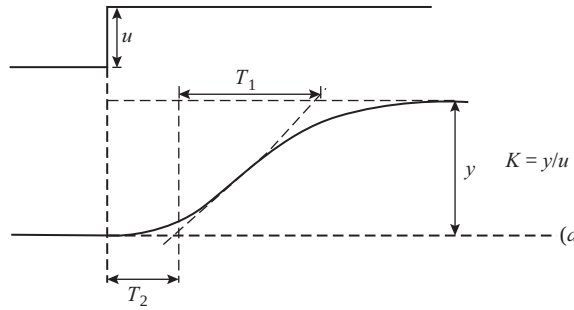


Figure 9.10 How the coefficients of equation (9.1) are determined

Notice carefully that these controller coefficients are suggested to achieve control of the very large class of processes that can be approximated by (9.1) and further that the aimed-for step response of the resulting closed loop system is underdamped with the characteristic that the magnitude of each overshoot/undershot shall be one quarter of the previous one. This type of response may not of course suit every application but the logic behind the choice is that such a response comes near to minimising the error criterion

$$J = \int_0^\infty |e(t)|dt$$

where $e(t)$ represents the error $y(t) - v(t)$ following the application of an input of a unit step to the input v at time $t = 0$.

Thus, the Ziegler–Nichols rules are an attempt to design an optimal controller for the unknown process.

Illustrative example

(Explanatory note: In real life, we shall rarely, except in simulation exercises, have a known, accurate, mathematical model of any actual plant that we might be called on to control. Often, we shall have to make do with a few response curves from which we will need to construct a data-driven model as an intermediate step in controller design.

In the example that follows, you are to suppose that we have obtained a step response of the plant to be controlled and, with the aid of that step response, will proceed to design a controller. How well will that controller work when it is implemented on the real plant? Normally, we would need to carry out plant trials to find out, but within this example, we are in a privileged position, because in this unrealistic case we do know the true transfer function model that generated the step response and will be able to see how the controller works on the 'true' plant.)

We choose as the process that is to be controlled a plant with true model

$$G(s) = \frac{4}{(s+1)(s+2)(s+4)} \tag{9.3}$$

but to be realistic, we don't (yet) allow ourselves access to knowledge of this model – only access to its response to a unit step (Figure 9.11(a)). From that figure and its amplification Figure 9.11(b), using the graphical construction given in Figure (9.10), we extract the approximate model

$$G'(s) = \frac{0.5e^{-0.4s}}{(1 + 2.12s)} \tag{9.4}$$

Figure 9.11(c) and (d) compare the actual response with the approximation. Then using (9.2), we find the three-term controller coefficients to be

$$\begin{aligned} &\text{Gain } C = \frac{1.2\ T_1}{K\ T_2} = \frac{(1.2)(2.12)}{(0.5)(0.4)} = 12.72 \\ &\text{Integraltimeconstant} = T_I = 2T_2/C = 0.0629 \\ &\text{Derivativetimeconstant} = T_D = 0.5CT_2 = 2.544 \end{aligned} \tag{9.5}$$

yielding the controller D as

$$D(s) = \frac{1}{s}(15.89 + 12.72s + 2.544s^2) \tag{9.6}$$

and the combination of controller and process in series as

$$G(s)D(s) = \frac{4(15.89 + 12.72s + 2.544s^2)}{s(s + 1)(s + 2)(s + 4)} \tag{9.7}$$

We have now allowed ourselves access to the true model $G(s)$ so that we can determine the step response of the closed loop, containing the three-term controller calculated via the approximation route.

The transfer function of the closed loop system $GD/(1 + GD)$ is

$$\frac{GD}{1 + GD} = \frac{4(15.89 + .12.72s + 2.544.s^2)}{s(s + 1)(s + 2)(s + 4) + 4(15.89 + 12.72s + 2.544s^2)} \tag{9.8}$$

To find an expression for the step response in the time domain of the closed loop system $GD/(1 + GD)$ shown above, we need to take the Inverse Laplace transform of $\{(1/s)(GD/1 + GD)\}$ as shown below

$$\begin{aligned} f(t) = {} & \mathcal{L}^{-1}\left(\frac{1}{s}\frac{GD}{1 + GD}\right) \\ & + \mathcal{L}^{-1}\left(\frac{1}{s}\frac{4(156.89 + 12.72s + 2.544s^2)}{s(s + 1)(s + 2)(s + 4) + 4(15.89 + 12.72s + 2.544s^2)}\right) \end{aligned} \tag{9.9}$$

If we use a Matlab command for the inversion in one sweep of the above transform, there is a danger of losing sight of the nature of the solution, so instead we factorise the expression and then take partial fractions to obtain

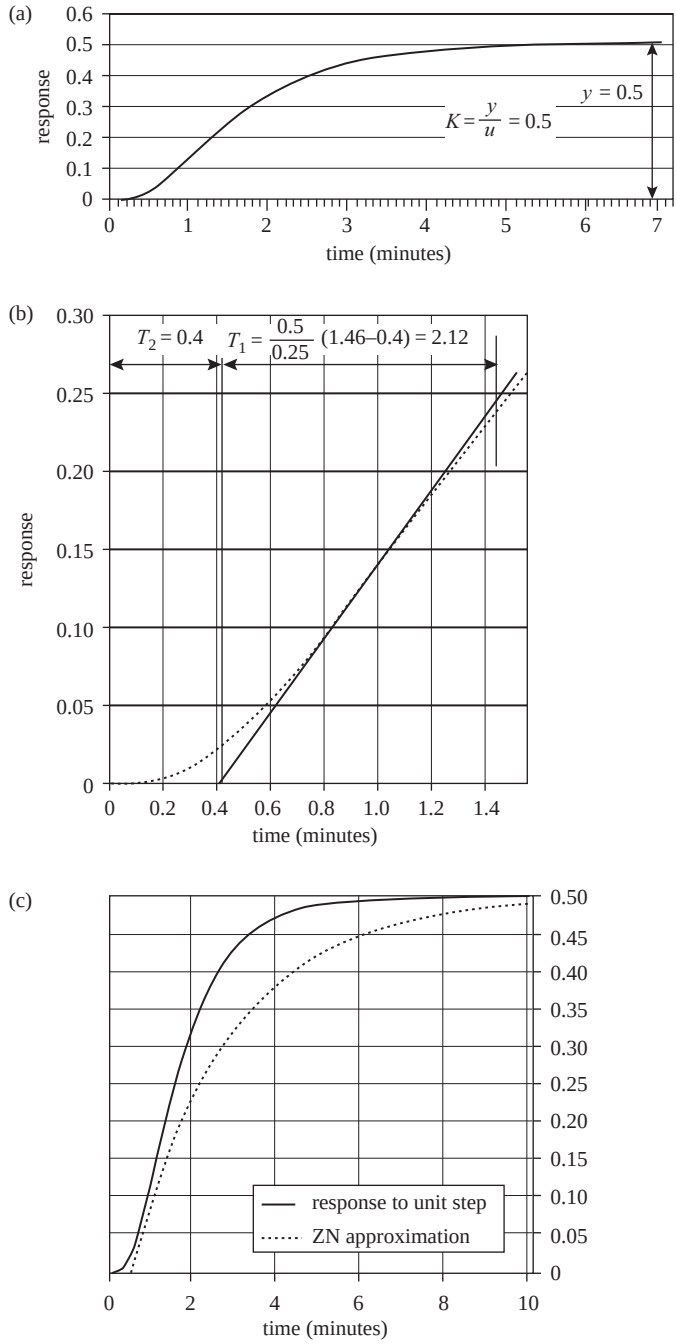


Figure 9.11 Continued

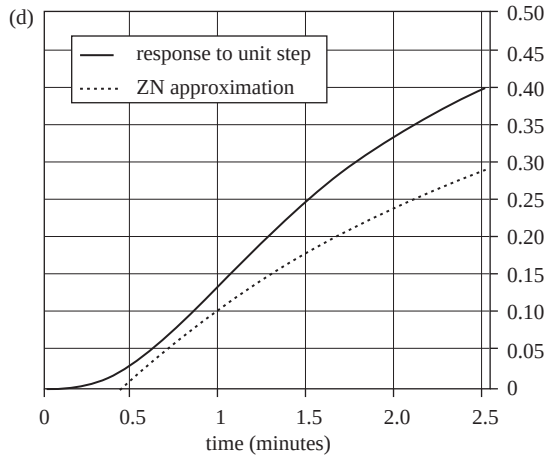


Figure 9.11 (a) Response of the process G(s) = 4/[(s + 1)(s + 2)(s + 4)] to a unit step input; (b) response to unit step (graph expanded near origin); (c) real process response and its approximation; (d) expanded detail from 9.11(c)

$$f(t) = \mathcal{L}^{-1}\left(\frac{0.995}{s} - \frac{0.04}{s+2.164} + \frac{0.125}{s+3.374} - \frac{1.079(s+1.17)}{s^2+1.478s+8.747}\right) \tag{9.10}$$

The last term has denominator with complex roots expressible as

$$(s + 0.739 + j2.864)(s + 0.739s - j2.864)$$

which can also be expressed as

$$(s + 0.739)^2 + (2.864)^2$$

still considering the last term in (9.10), we note that it has the form

$$\frac{1.079(s+a)}{(s+b)^2+\omega^2}$$

which has the inverse transform

$$f(t) = \frac{1}{\omega}\sqrt{(a-b)^2+\omega^2}\quad e^{-bt}\sin(\omega\, t + \phi)$$

where

$$\phi = \tan^{-1}\left(\frac{\omega}{a-b}\right)$$

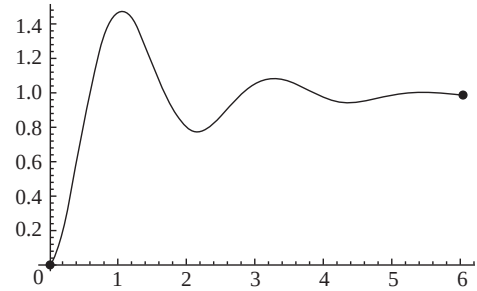


Figure 9.12 Step response of the closed loop system G(s)D(s)/(1 + G(s)D(s))

and in our case the time function corresponding to the complex term is therefore

$$1.079(0.35)(2.896)\ \exp(-0.739t)\ \sin(2.864t + 1.42)$$

$$= 1.093\ \exp(-0.739t)\ \sin(2.864t + 1.42)$$

and the time function $f(t)$ corresponding with (9.10) can now be written as

$$f(t) = 0.995 - 0.04e^{-2.164t} + 0.125e^{-3.374t} + 1.093e^{-0.739t}\sin(2.864t + 1.42) \tag{9.11}$$

It is easy to see that the response will be dominated by the sinusoidal term in its envelope of decay and this is confirmed in the plot of Figure 9.12. It is clear that a good closed loop response meeting the criteria outlined above has been obtained with little effort using only information form a single step test at the process.

9A How to learn something from the first part of a step response

The initial part of a step response gives information about the order of the process (Figure 9.13). For a first-order system, the steepest part of the response is at the origin but for higher order processes the response clings to the time axis before rising. To understand this, let A, B be first- and second-order processes, respectively, and let a, b, c be process parameters with obvious meanings, then the respective step responses are:

$$f_A(t) = (1 - e^{at}), \qquad f_B(t) = \frac{1}{bc}\left(1 + \frac{1}{b-c}(ce^{-bt} - be^{-ct})\right)$$

and the derivatives are

$$f_A'(t) = ae^{-at} \text{ and } f_A'(0) = a$$

and this value a, the inverse of the process time constant, represents the steepest part of the response curve

$$f'_B(t) = \frac{1}{bc(b-c)}(bce^{-ct} - bce^{-bt}) = \frac{1}{(b-c)}(e^{-ct} - e^{-bt})$$

It is clear that the initial part of the step response of second-order process B has zero slope, since the second term in the expression for the derivative is zero at $t = 0$.

The step response of a linear process and its frequency response both contain exactly the same information and both can be considered to be non-parametric models of the process (as opposed to transfer function models which have an **orde**r and contain **parameters** whose numerical values need to be chosen.

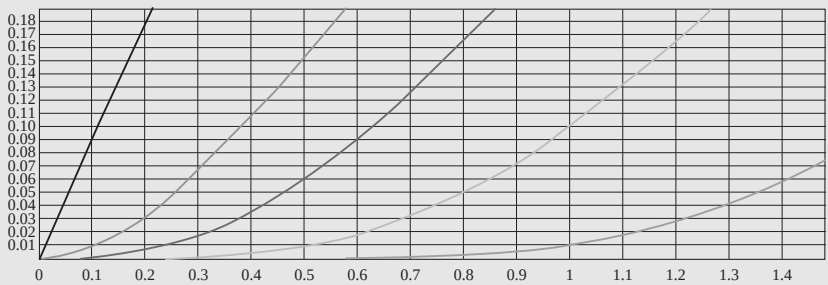


Figure 9.13 High-order step responses cling to the axis! Comparison of the initial parts of unit step responses for first-, second-, third-, fifth- and tenth-order processes all with unity steady state gains

9.3.4.2 To fit the controller into a closed loop with the process to be controlled and go through a tuning procedure online

The method is more difficult to conduct on a real plant since it first requires that the controller with integral and derivative actions disabled be fitted into closed loop with the process. The controller gain C must then be increased until the loop oscillates continuously at a constant amplitude. (This is not so easy as it sounds!) The controller gain C^* that causes continuous oscillation of the loop and the period T^* of the resulting oscillation are noted. From these two pieces of information, the three-term controller coefficients can again be determined from (additional) Ziegler–Nichols (1942) rules as follows:

$$\begin{aligned} &\text{Controller gain } C = 0.6C* \\ &\text{Integral time constant } T_I = 0.5\ T* \\ &\text{Derivative time constant } T_D = 0.125T* \end{aligned} \tag{9.12}$$

An exercise for the reader to compare the two tuning methods: Starting with $G(s)$ as given in (9.3), devise and apply any theoretical method to determine by any method

C^* and T^* as described in this section. Calculate the controller coefficients using (9.12). Compare with the controller coefficients found above in Section 3.4.1. Comment constructively.

9.3.4.3 To fit a so-called self-tuning controller into closed loop with the process. After a learning period, the controller will hopefully have chosen its own coefficients

There are quite a number of self-tuning algorithms, many of them quite complex. Some approaches use an expert system that emulates a skilled human control engineer, other approaches emulate approach 9.3.4.2, exciting the loop and then interpreting the responses. Every practical self-tuning algorithm must necessarily have some sort of confidence test to pass before it can be allowed to implement its choice of coefficients onto the real process. There is an extensive literature.

A paper describing a method for data-driven tuning of a three-term controller without any use of a model is Keel (2008).

9B New York to San Francisco telephony: an early illustration of the spectacular success of feedback in achieving high-fidelity amplifications of signals

In early long distance telephony, messages travelled along a land line with repeater stations (audio frequency amplifiers) at intervals to boost the signal strength.

Early electronic amplifiers were highly sensitive to variations in thermionic valve (USA tube) characteristics and variations in supply voltage. This meant that the gains were not constant and that consistent high-fidelity amplification was not possible. If, say, ten such amplifiers each reproducing a signal with 90% fidelity were connected in series (as repeater stations must be), then the fidelity of the overall line would be $100.(0.9)^{10} = 35\%$. Because of the poor robustness of available repeater amplifiers it was decided that no more than six such repeaters could be tolerated along the whole 3000 mile (4800 km) line. The signal strength was kept high by the use of massive power cable capable of carrying 50 A and weighing half a ton per mile (300 kg/km).

No doubt motivated by this problem, Hendrik Bode, Bell Telephone Laboratories, *c.* 1927, invented and implemented feedback amplifiers to produce highly insensitive (i.e. gain robust to parameter changes) amplifiers for transcontinental telephony.

These amplifiers using feedback were of such high fidelity that 600 could be used sequentially as repeater stations when a new New York to San Francisco light weight cable was laid in 1941.

Figures 9.14–9.16 illustrate this example.

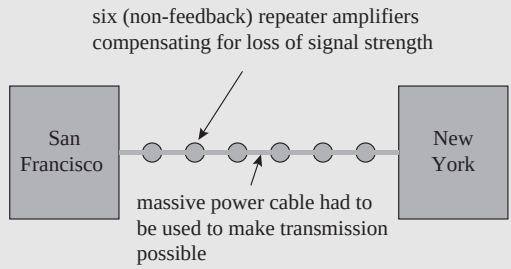


Figure 9.14 First trans-US telephone cable. No more than six amplifiers could be used because of the cumulative distortion effect

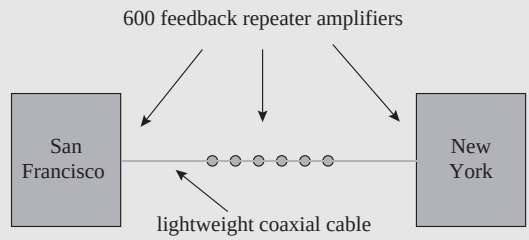


Figure 9.15 By 1941, the availability of Bode's feedback amplifier allowed 600 amplifiers to be connected sequentially and a low-cost lightweight cable to be used for the connection

$$y = \frac{KG}{1 + KG}\,u \qquad \text{nominal process}$$

$$y = \frac{K(G + \Delta G)}{1 + K\,(G + \Delta G)}\,u \qquad \text{perturbed process}$$

Figure 9.16 If the amplifier gain K *is sufficiently high, the feedback loop is insensitive to process perturbations* ΔG *or gain perturbations* ΔK

9.4 Control systems for batch process

The essential difference between a batch and a continuous process (both making the same identical product) might be looked at simplistically but usefully as in Figure 9.17.

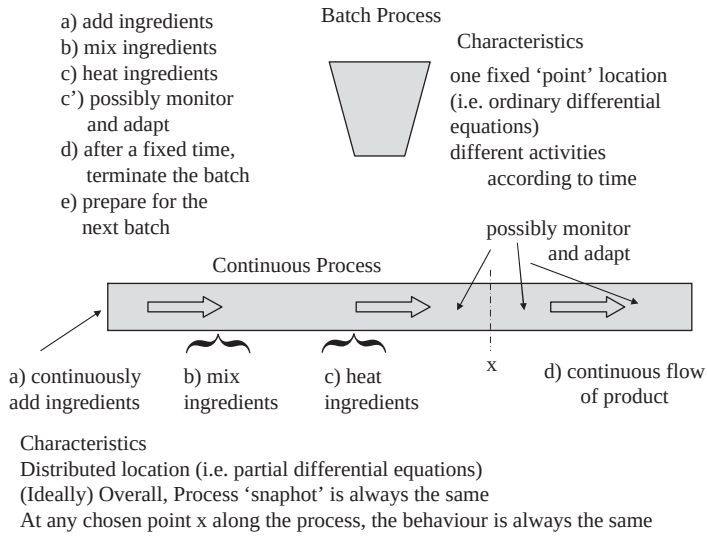


Figure 9.17 Schematic diagram emphasising the structural differences between batch and continuous processes

From the figure, it becomes clear that the control of a batch process involves at least three stages:

(1) Set the initial conditions.
(2) 'Launch' the batch along a desired (optimal (an additional problem compared with a continuous process: how determined?)) trajectory, specified, for instance in terms of temperature (or in general a vector of variables) against time.
(3) Possibly monitor the progress of the batch and ensure that the trajectory is followed or (more sophisticated) modify the desired trajectory as necessary to maintain its optimality.)
(4) Terminate the batch at a time chosen to maximise profitability. (That time may be influenced by scheduling factors including those of downstream processes.)
(5) Possibly based on experience from the current batch n, modify the initial conditions or trajectory for batch $n + 1$.)

It is clear that the control of batch processes involves more challenges than the control of continuous processes, yet control theory in general concentrates its attention on the continuous process.

In reality there are relatively few processes that are really continuous. Nearly all have difficult start-up and end-off situations (think take-off and landing!) to contend with as well as step disturbances (raw material change or order change).

Control systems design for batch processes: should one design for the worst case?

Except when safety issues are involved, it is rarely economic to design a system to behave normally even when very rare, very large disturbances present themselves. A good pragmatic approach invented by a colleague, John D. Gifford, is to examine the challenges represented by a large number of previous batches and to choose a single 'difficult, but not untypical' incoming disturbance profile and to design controls that deal satisfactorily with that. More difficult, very rare, untypical profiles will not be fully corrected, it being considered uneconomic to over-engineer the system for those few cases.

A related interesting contribution has been made by Lane (2005) with his mathematically defined concept of 'Approximately worst' situations that can be used as part of design specifications.

A great deal could be written about the scheduling of batch processes, but here we will mention just two of the chief issues.

Many companies operate a group of (say) six to ten batch reactors all making the same product. Batch operations often have to be planned so that the flow of semi-product fits into the pattern of downstream flowline processes.

Considering just a single batch process, there is frequently the issue, for how much longer to run batch n. Often, by running longer we can extract more value from the ingredients but we may produce less product overall, since we are delaying the next run, $n + 1$, of our process.

Of more interest to control engineers is batch-to-batch adaptation and in-batch control to improve product yield and batch efficiency. In-batch control involves some very interesting techniques, possibly involving trajectory following with local linearisation along a trajectory, and within-batch updating of the desired trajectory.

One of the simplest possible approaches to batch-to-batch adaptation is, in principle, as follows:

A batch process converts an initial condition in state $x(t_0)$ into a final condition with state $x(t_f)$, hopefully satisfying $x(t_f) = x_d$, where x_d is a desired product specification state. We can write

$$x(t_f) = f(x(t_0), u(\tau), \tau \in [t_0, t_f]$$

$u(\tau)$ being a control policy on interval $[t_0, t_f]$.

After several well-behaved similar runs of the batch process, it should prove possible to estimate the sensitivity coefficient

$$\frac{\partial x(t_f)}{\partial x(t_0)}$$

and with some ingenuity

$$\frac{\partial x(t_f)}{\partial u(\tau)}, \tau \in \left[t_f, t_f\right]$$

and to use these to attempt batch-to-batch improvements, using rules such as

$$x(t_0)_{n+1} = x(t_0)_n + \frac{\partial x(t_f)}{\partial x(t_0)} \Delta x(t_f)$$

$$\text{where } \Delta x(t_f)_n = x_{d-} - \partial x(t_f)_n$$

(Several case histories involving control of batch processes will be found in Appendix A.)

9C The idea of a probing controller

Akesson and Hagander (2000) have proposed a so-called probing controller that uses a generic idea for tracking just below invisible varying and unknown constraints that occur in a batch process. The idea is to make probing pulses in the glucose feed rate and to monitor the responses that change as the constraint is approached. By this method, it is possible to detect and avoid a characteristic saturation linked to undesirable by-product formation. Figure 9.18 shows how in *Escherichia coli* fermentations, the optimal carbon feed rate will run along invisible constraints. The probing controller finds these boundaries by pulsing the feed rate as shown in Figure 9.19 and observing the nature of the response.

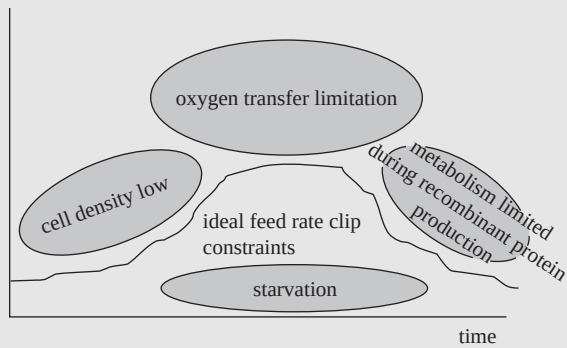


Figure 9.18 Carbon feed rate constraints in Escherichia coli*–based expression systems. The trajectory should be as close as possible to the three upper (invisible) constraints)*

The idea could be adapted to other processes where variable invisible constraints have to be approached as closely as possible.

See Akesson M., Hagander P. 'A simplified probing controller for glucose feeding in Escherichia coli cultivations'. *Proceedings of the IEEE Conference on Decision and Control*; Sydney, 2000, vol. 5, pp. 4520–5

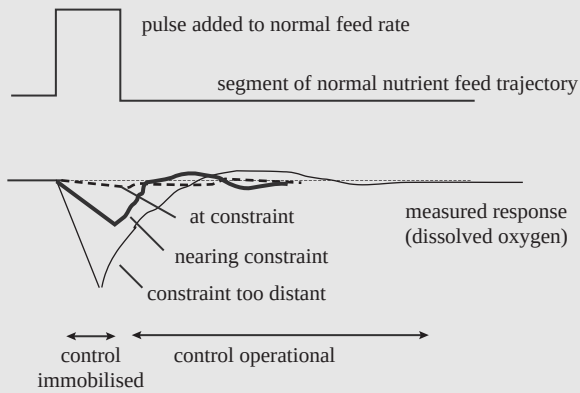


Figure 9.19 How the nearness to the constraint can be inferred from the measured responses to the injected pulses

9.5 Input shaping

Input shaping refers to a control method in which a specific transient behaviour is obtained from a system by the input of a specially synthesised input signal. Quite frequently, input shaping strategies are used to change the position of a highly oscillatory mechanical system quickly with minimum transient oscillation. Professor Patrick Parks once demonstrated the technique very effectively with a quite sizable rope, alongside his talk 'How to shake a piece of string straight' (Bell (1972)), on a conference platform.

For a more industrial application, consider (Figure 9.20) the idealised overhead travelling crane used extensively in heavy industry and for handling containers. It runs on girders and can travel in two dimensions but, for simplicity, is restricted to one dimension in our diagram. This diagram shows three stages in the typical movement of a load.

Astute crane drivers have, of course, learned by experience how to minimise the time-consuming load oscillations by cleverly modifying the pattern of how to move the crane along the track. Probably they cannot quantify exactly what their strategy is, but in control system parlance, it is called **input shaping.**

Although very simple indeed, it can be a remarkably effective strategy.

We shall here only illustrate an important general principle – how to cancel out the inevitable oscillation that must occur in this type of application and then show how to progress to input shaping (Figure 9.21).

The very readable paper Singh (2010) should be consulted to see how input-shaping techniques can be developed and applied in practice. The paper is strong on practical robustness issues; examining and quantifying how modelling errors will convert into residual, uncompensated oscillations.

The transient oscillation from the initial impulse can be cancelled completely after the first half cycle by inputting an exactly timed anti-phase impulse of just the right amplitude.

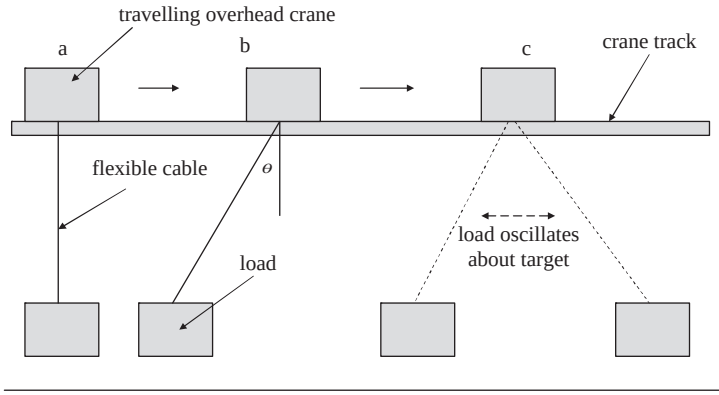


Figure 9.20 An idealised overhead travelling crane. (a) The load is lifted vertically off the ground; (b) the crane travels at a uniform rate towards the target position with the load following, left behind by angle θ because of load inertia; (c) the crane stops at the target position, the load arrives later and then oscillates pendulum-like about an eventual equilibrium point

In order to be of any practical use, signals such as steps need to be used, for example to move a load by crane from one location to another.

It is explained in Singh (2010) how the two-impulse sequence that cancels oscillation after the first half cycle may be convolved with any other desired command signal and the convolution product so arrived at will then (ideally) carry out a designated task with oscillation cancellation. Figure 9.22 demonstrates this command signal

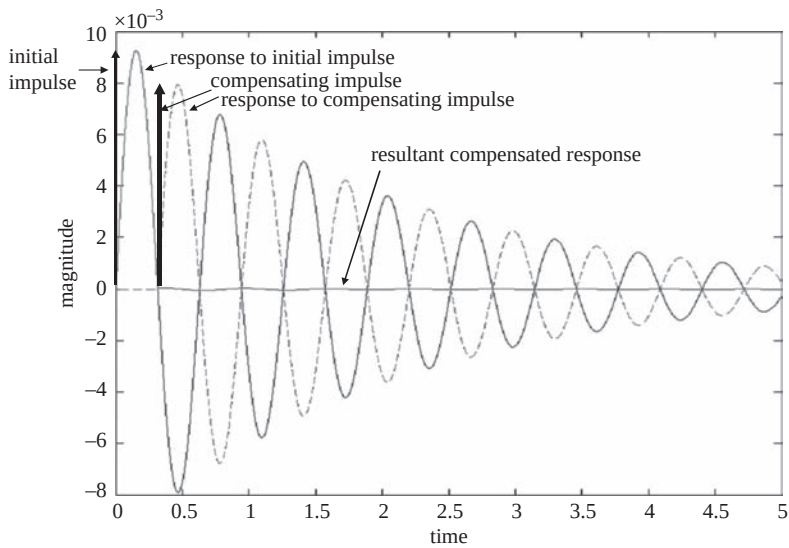


Figure 9.21 The principle of input shaping

synthesis for an undamped load. In this simple example, a finite length step is convolved with a two-impulse sequence to produce a stepped command signal.

Note: The convolution operation, denoted (*), of two time signals $g(t)$, $h(t)$, is defined as

$$g(t)^{*}h(t) \triangleq \int_{-\infty}^{\infty} g(t-\tau)h(\tau)d\tau$$

and this operation is frequently carried out graphically by reflection in the y axis and translation of one of the functions, followed by graphical multiplication of the two resulting functions. For the simple case of an undamped load of period 2Δ it can be seen, a shown in Figure 9.22 that the convolution of the long rectangular command signal with the two-impulse oscillation-cancellation duo results in a stepped oscillation-cancellation command signal. [Note that the operation of convolution is much easier in the s domain for, let g, h have transforms $G(s)$, $H(s)$ then convolution is simply performed by transform multiplication since convolution in the time domain is equivalent to transform multiplication $G(s)$, $H(s)$ in the s domain].

9.6 Gain scheduling (to allow a control system to operate successfully when the process to be controlled changes its characteristics over so wide a range that no constant controller can be found that performs adequately)

Gain scheduling is the name given to a large group of techniques, ranging from the empirical (in which a current control algorithm is somehow chosen and implemented from a set of stored algorithms) to the sophisticated (in which controller

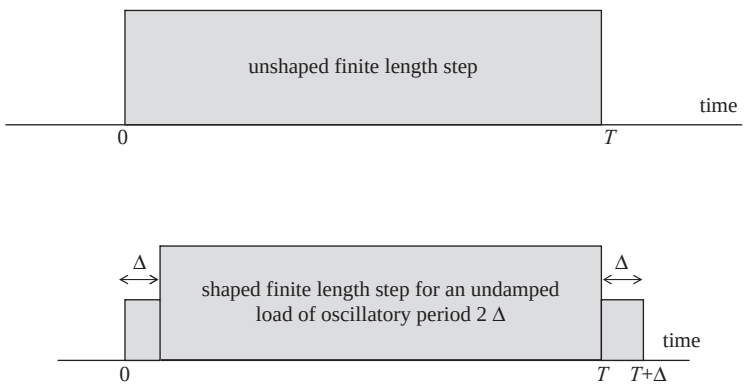


Figure 9.22 (Top) The unshaped step signal that will move the load as required but with swinging oscillation; (bottom) how an oscillation-cancelling command signal may be shaped by convoluting two impulses separated by half a period with the original step required by the application

parameters may be pre-calculated and implemented as a priori functions of the process state).

Some processes are so severely non-linear that no single linear controller with constant parameters can yield a satisfactory stable performance as the operating point of the process ranges over the non-linearity. If the non-linearity is smooth, known and constant over time, then a small number of linearised models of the process, say G_1 to G_n and a set of linear controllers D_1 to G_n could be arranged in a gain scheduling strategy to be switched in to control the process as the process operating point ranges through the approximate models G_1 to G_n. The reader will probably have guessed that the hard part of this approach is to find a practical and reliable switching strategy that does not produce spurious and possibly destabilising transients of its own making.

The more sophisticated approaches to gain scheduling merge into the topic of adaptivity, learning and robustness. All those topics are concerned in essence with attempting to obtain maximum system performance while guaranteeing stability in the presence of noise, significant process variability and uncertainty.

Because of this merging of approaches, gain scheduling will be discussed again with adaptivity, learning and robustness in Chapter 16.

9.6.1 Gain scheduling: traditional ad hoc and modern linear parameter varying (LPV) based

Gain scheduling is the name given to a group of strategies that aim to provide reliable pragmatic adaptivity for systems whose characteristics change significantly over a range that no single robust controller can span satisfactorily.

Gain scheduled systems are open loop pre-programmed in that their controller gains (and possibly other parameters) are directly linked with an online measurable variable, such as airspeed in an aeronautical application, or with operating point. Gain scheduled systems differ from adaptive systems in that they take no account of the actual system response and are essentially open loop rule-based. Their behaviour, in a given set of circumstances, is always reassuringly, reliably and predictably the same.

9.6.2 LPV gain scheduling as a step forward from traditional gain scheduling

If there is such a thing as a traditional approach to gain scheduling (Figure 9.23), then it is performed roughly as follows:

Given a non-linear process

$$\dot{x} = f(x, u), y = Cx$$

where the non-linearity is assumed smooth,

(1) Choose a number of operating points $x_{op}(1), .., x_{op}(k)$ that together span the expected operating range (in state space) of the proposed system when under control.

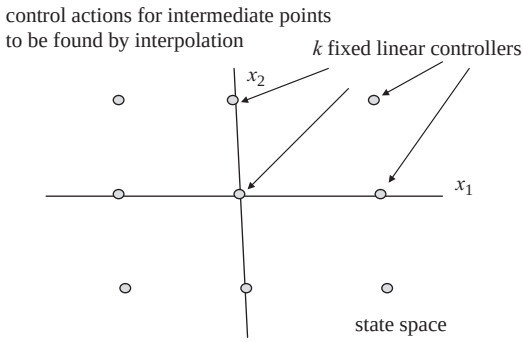


Figure 9.23 Traditional gain scheduling shown symbolically. At all times the control action to be implemented is found by interpolating between the actions of the pre-calculated controllers (It being assumed that the current state of the process can be measured at all times)

(2) At each operating point, linearise the process model to yield

$$\dot{x} = \frac{\partial f}{\partial x}(x_{op}(j), u_{op}(j)) + \frac{\partial f}{\partial u}(x_{op}(j), u_{op}(j))$$

Each of these k models is linear and time invariant.

(3) For each of the k linear models, design by usual feedback control methods one feedback controller $L(j)$, $j = 1, \ldots, k$.

(4) In use, measure the state x and continually interpolate in the 'controller space' to determine the currently required control action.

Gain scheduling techniques have traditionally been a rather ad hoc set of approaches, driven more by industrial needs than by academic researchers.

More recently, however, LPV approaches to gain scheduling have begun to offer more theoretically sound, yet industrially applicable algorithms: they offer the possibility of a single smooth parameter-varying model, representing continuously a non-linear process over its whole operational range.

A further advantageous possibility is to use the single parameter-varying model to design a single parameter-varying to cover an agreed range of operation of the process.

There are some requirements that need to be met before LPV gain scheduling can be applicable. The LPV controller must have constant access to the process state and there must be no significant discontinuities in the process behaviour over the chosen LPV region. Also, there will naturally be an application preference for processes where the state-dependent parameter changes only slowly.

Finally, note that quite a number of industrial processes and aerospace applications change characteristics during operation to such an extent that switching of controllers, rather than smooth transitions, is required.

A good example is in batch steelmaking in large electric arc furnaces. During the initial melting of 100% scrap, a short, thick arc with high current and low voltage is used and automatic positioning of the electrodes keeps the gap between

electrode tips and molten bath controlled so that the power input meets a specified level. At a later refining stage in the process, a long, thin arc is required with a high voltage and low current. Because the operating point on the highly non-linear arc characteristic reverses its slope during this refining stage (compared with during the melting stage), it is necessary automatically and routinely to insert a negative sign into the electrode positioning feedback loop when changing between melting and refining stages.

9.6.3 Outline of the LPV approach to gain scheduling

Given a process described by the non-linear smooth equation

$$\dot{x} = f(x, u)$$

we seek a LPV model of the form

$$\dot{x} = A(p)x + B(p)u,\ p \in \Omega$$

where Ω is the 'box' over which the state-dependent parameter $p(x)$, which needs to be measurable online, is allowed to range.

Once a single LPV model of the process is available, the aim is to design a single LPV controller that can control the non-linear process over the chosen box of validity with guaranteed stability. This is not too difficult and many approaches have been suggested in the literature. As is usual, obtaining a process model of sufficient accuracy over the necessary range of operation will be the hardest step in the procedure. For this reason, there is considerable interest in methods for obtaining LPV models of real processes from measured process data. See Laurain (2011) and also Ali (2010).

Matlab provides a command ***hinfgs*** for the synthesis of gain scheduled H^∞ controllers.

The user must provide a parameter-dependent process model, whose time-varying parameter moves within a specified box Ω. The command returns a parameter-dependent feedback controller $K(p)$ that minimises an appropriate quadratic H^∞ performance index.

9.7 Converting a user's requirements into a control specification

A user's requirement will usually be application specific (keeping a ship on a desired course to a particular accuracy; dispensing a certain weight of soap powder; neutralising a liquid effluent before discharge to a river; maximising the yield of pharmaceutical product from a given batch of raw material etc.).

An unrealistic (oversimplistic) conversion of the user's requirement into a control specification, against which the system will be built, will result in the

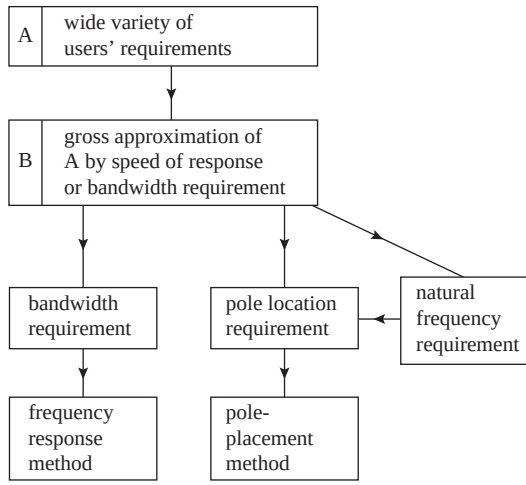


Figure 9.24 Alternative design approaches

building of an unsatisfactory system. This aspect (conversion of a user's requirements into a specification) is frequently a weakness in the control design chain.

Let us switch our thoughts temporarily to the amount of freedom that a designer has in designing a simple control loop. First, the control loop will need to be stable with a reasonable stability margin. This stability margin will need to be more or less the same, regardless of the application; hence, although the designer has to fix the stability margin, that margin will be virtually the same regardless of application and therefore this aspect cannot be regarded as a variable design parameter. The other variable that can be fixed by the control designer is the *speed of response* or the closely related parameter, *system bandwidth*. Both of these quantities are related in a well-defined way with *pole locations* and with *system natural frequency*.

Thus, in the design of a simple control loop, the designer will often be seeking to achieve a particular bandwidth or a particular speed of response by fixing pole locations, by fixing natural frequency or by fixing bandwidth in the system to be synthesised. Figure 9.24 illustrates the design route.

Two important questions arise:

Question 1: How can diverse user's requirements be converted into very simple speed of response or bandwidth specifications?

Answer 1: They can't, except in a small minority of cases that are mostly confined to the servomechanism field. In most other cases, the designer spends huge proportions of his time coping with application-dependent problems, using general engineering knowledge and ad hoc methods.

Question 2: What sets an upper limit on the speed of response (or bandwidth) that can be obtained in a particular application?

Answer 2: Very interesting! In linear control theory, there are by definition no upper limits on anything. Thus, linear control theory can produce a system that will turn a supertanker onto a new course in microseconds or less, provided that the linearity is not violated.

Thus – the upper limits on performance are set by factors that do not appear at all in the design process – clearly this is very unsatisfactory!

In practice, the designer must choose, for example, an electric motor to give the acceleration that he or she needs. As larger and larger motors are considered, so the acceleration will approach that given by an unloaded motor. If this acceleration does not meet the specification, another approach must be found. The point to note is that none of this procedure is part of the control design procedure but is injected by the designer in what is usually called engineering interaction with the design process!

9.8 Methodologies for deciding the scope and attributes of automatic control schemes for industrial application (in particular, methodologies for economic justification of investment in automation)

For a control engineer working in industry, economic justification is not an optional extra. Improved control implies better and more powerful actuators and the purchase of expensive novel sensors. No well-managed company will sanction the necessary expenditure without a water-tight justification that each increment of expenditure will lead to sufficient payback.

9.8.1 Methodologies and illustrations

Given a set of interlinked industrial processes that together constitute a plant producing some product from incoming raw materials, control theory and practice will tell what might be achieved at each of the processes. The list of all possible schemes that might be designed would be formidable indeed. The question we want to consider here is, given a particular industrial configuration, how can one describe on the scope, configuration and functionality of appropriate control systems to be integrated into the manufacturing facility in something close to an optimal way.

Here we review some of the available methodologies but it has to be said that there is a distinct shortage of methodologies – in fact most of those described below were originated by the author. The lack of literature is a sign not of lack of importance of the approaches but rather a result of the methods being unglamorous and theoretically undemanding, making them unattractive to academics because of their unsuitability for publication.

The first suggestion is to define for a whole production sequence a broad sweep performance index of the form

J = APP (price at which one tonne of product sells − cost of manufacturing one tonne of product)

where APP is the annual production, in tonnes, of prime product.

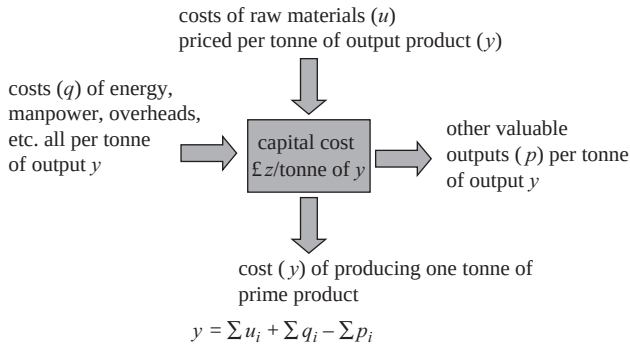


Figure 9.25 Calculation of the cost (y) of production for an entire plant or for a single process in most of my work z has been omitted

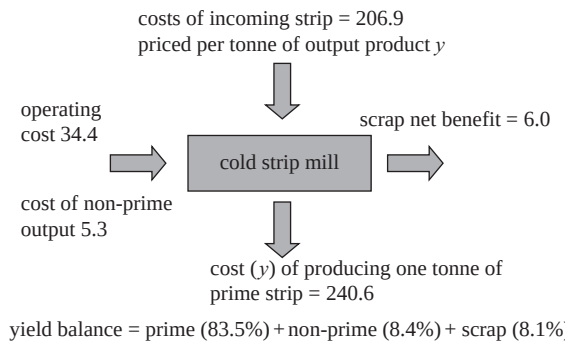


Figure 9.26 Sample cost calculation: cold strip mill (strip from strip production)

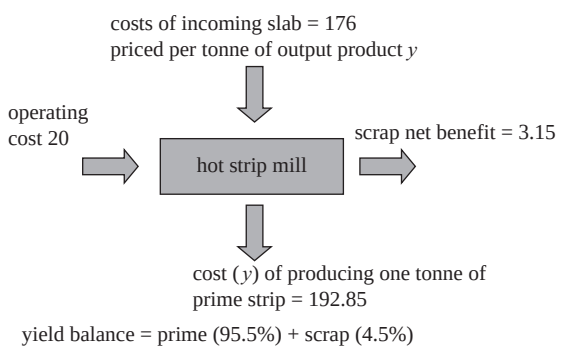


Figure 9.27 Sample cost calculation: hot strip mill (strip from slab production)

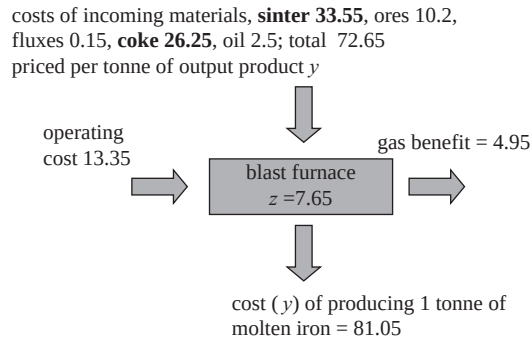


Figure 9.28 Sample cost calculation: iron-making (molten iron from sinter production)

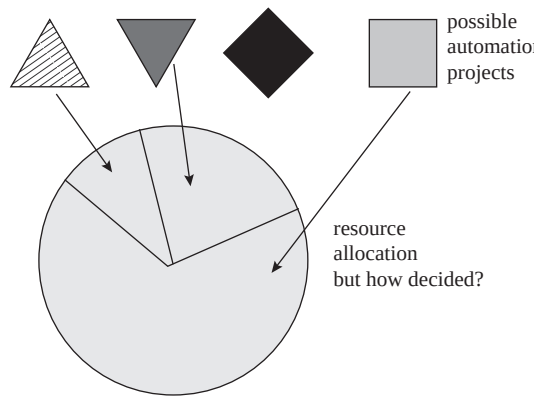


Figure 9.29 How do we choose automation projects and what should be the resource allocation for each?

Our broad aim in choosing between alternative strategies will then be to maximise *J*, but how do we calculate the cost of manufacturing one tonne of product. The solution is to develop a model of the form shown in Figure 9.25 for every process in the production sequence and eventually through the use of these interconnecting models we can link right back from product leaving the factory to raw materials entering the factory. The operation of the models is self-explanatory but it remains to mention that the models have to be parametrised by analysing masses of real industrial data. The examples given here as Figures 9.26–9.28 relate to the steel industry and show how the product, steel strip, links back to the basic raw materials of iron ore and coking coal. The figures given here are realistic but they have been modified for confidentiality reasons. The models allow the economic context of the

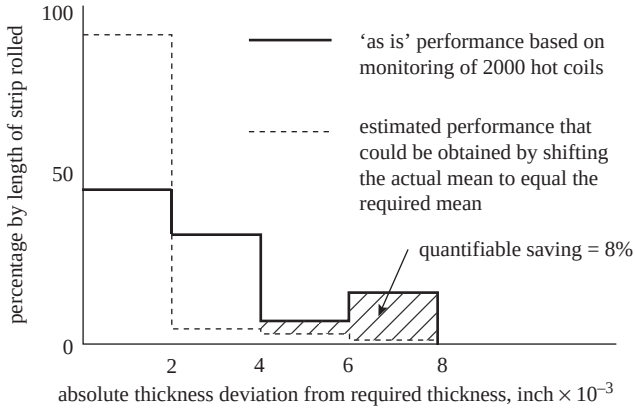


Figure 9.30 Justification histogram

process to be understood with the main areas for possible savings being visible to a large extent by inspection.

Figure 9.30 shows a 'justification histogram' produced by the author with colleagues from measurements on 2000 batches of steel strip. It shows that almost 10% of the lengths of strip produced were outside the allowed thickness tolerance and allows quantification to be made of the benefits of tighter control.

Figures 9.31 and 9.32 show what I call an 'Economic Dynamic Programming' approach to choosing the best control configuration for a set of closely interlinked sequential processes. The idea is that, at each stage of the process, there are, in the example, three control design choices – let us say – 'minimum cost', 'medium

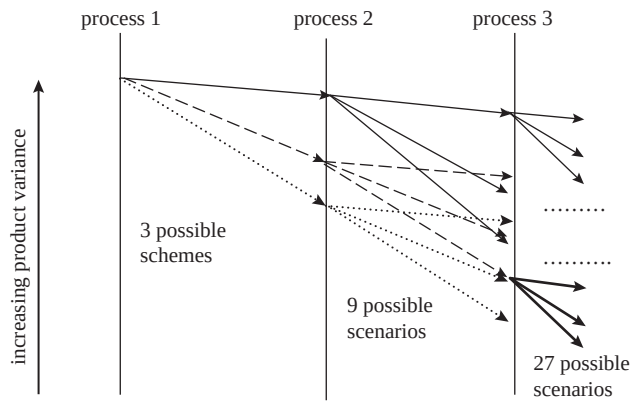


Figure 9.31 Investment strategy tool for n linked processes (author has used this tool with a dynamic programming approach to eliminate all definitely sub-optimal strategies)

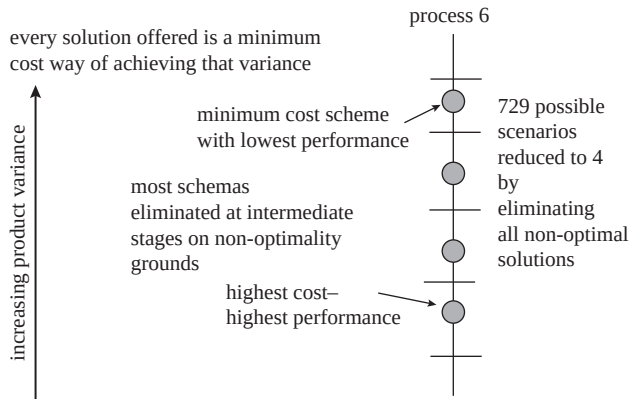


Figure 9.32 Investment strategy tool for six processes with three choices at each stage

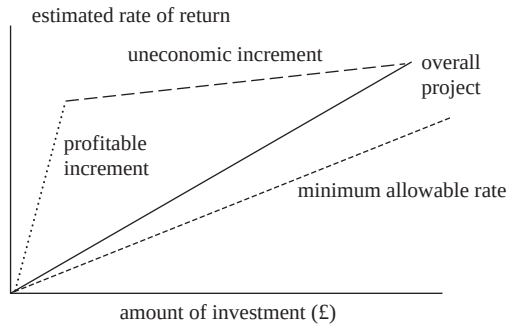


Figure 9.33 How an overall project may contain uneconomic increments

cost', 'high cost state of the art'. This means that, in a six-stage process, there are $3^6 = 729$ possible configurations.

The assumed aim of the control system in this simple example is to reduce product variance, and the dynamic programming approach eliminates all non-optimal ways of achieving a particular variance so that, by coarse discretisation, we can obtain, as shown in Figure 9.32, four possible levels of performance, and for each we offer the unique minimum cost way of achieving that performance.

(For each of the three possible solutions we have an implementation cost and of course we need either a deterministic or stochastic simulation that can generate estimates of the intermediate performances.) The method allows the designer to allocate the task of reduction of variance optimally between several closely linked sequential process stages.

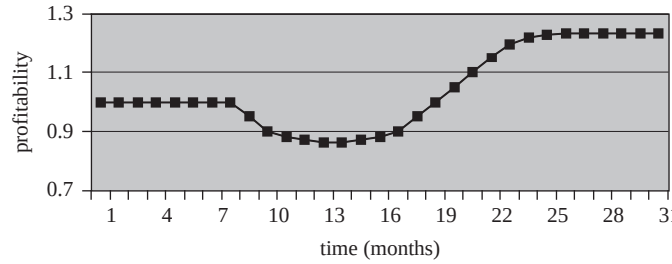


Figure 9.34 Typical time to obtain project benefits for a major project

In calculating the rate of return for a possible automation scheme, there will usually be a lowest acceptable rate of return, dotted in Figure 9.33, and all schemes, to receive funding, must normally generate a return at a slope greater than this. Note though that most automation schemes can be broken up into several component parts (Figure 9.33) and that as shown in the figure unprofitable components may be hidden by the compiler of the diagram.

Figure 9.34 shows a typical time history for the increase in performance for the commissioning of a typical large and complex automation scheme. The characteristic performance fall before rising degrades the return on capital very significantly and may make a whole automation project uneconomic.

Finally, Figure 9.35 shows how, for many processes, there is another technico-economic consideration – how to decide on an optimal throughput rate that is a compromise between high yield and high throughput. Such problems arise across a wide range of applications from pharmaceuticals – where pushing production will usually lower yields from the expensive raw materials – to the scheduling of the speed for a supertanker carrying oil over several thousand miles – where high steaming speeds get the oil to market earlier but use a disproportionate amount of extra fuel in doing so. For all these cases, a market-dependant operating point, shown by the asterisk in Figure 9.35, needs to be chosen as yet another economic aspect of practical control.

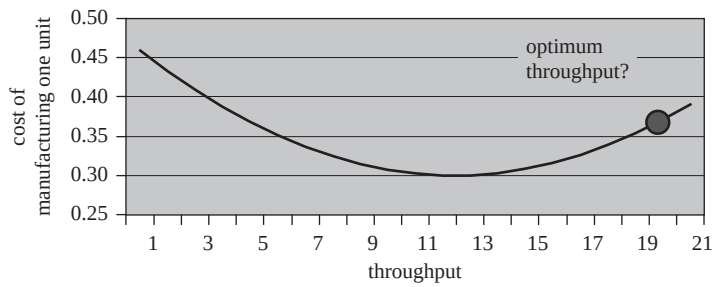


Figure 9.35 Matching throughput to market conditions

References on economic justification of investment in automation

Allen D. *Economic Evaluation of Projects* (London, Institute of Chemical Engineering, 1990)

Anderson J.S., Brisk M.L. 'Estimating the benefits of advanced process control'. *Proceedings of IChemE symposium on Advances in Process Control*; York, 1992

Baldwin B. 'Economic justification of process improvement and automation projects'. *IEEE Annual Textile, Fiber and Film Industry Technical Conference*; Greenville, S.C., 1997, pp. 1–5

Bauer M., Craig I.K. 'Economic assessment of advanced process control – A survey and framework'. *Journal of Process Control*. 2008;**18**(1):2–18

Brennan D.J. *Process Industry Economics* (London, Institute of Chemical Engineering, 1997)

Brisk M.L. 'Process control: theories and profits'. *Proceedings of IFAC Congress*; Sydney, 1993, vol. 7, pp. 241–50

Ceroni J.A. *Economic Rationalization of Automation Projects*. Springer Handbook of Automation, Part E (New York, Springer, 2009) pp. 699–713

Craig I.K., Henning R.G.D. 'Evaluation of advanced industrial control projects: a framework for determining economic benefits'. *Control Engineering Practice*. 2000;**8**(7):769–80

Marlin T.E., Brisk M.L., Barton G.W., Perkins J.D. 'Experiences from an industry-university study on control benefits'. Preprint of Conference on Control Systems 2000, Quantifying the Benefits of Process Control; May 2000, pp. 5–9

Zhao C., Zhao Y., Su H., Huang B.'Economic performance assessment of advanced process control with LQG benchmarking'. *Journal of Process Control.*2008;**19**(4):557–69 [*An optimisation-based approach for quantifying the potential for improved profitability obtainable through control system improvements*.]

Note: Further source material and suggestions for further reading to support the earlier topics of this chapter will be found in Chapter 18.

Chapter 10

Discrete time and digital control systems

10.1 Introduction

Many of the problems that were created in the early days of using online computers for closed loop control have largely disappeared or at least become de-emphasised by the ready availability of ever-increasing computer power and computer speeds.

It nevertheless remains true that, despite the many well-publicised practical advantages of digital control systems over their analogue equivalents, the sampling on which they depend does introduce inevitable performance degradation, as a comparison of Figures 10.2 and 10.4 in this chapter demonstrates. (Note also how the time signal from a digital radio lags that from an analogue device.)

In summary

(i) the delays introduced into a feedback loop by sampling are necessarily destabilising; and
(ii) very rapid sampling of noisy continuous signals can amplify the noise content, particularly if differentiation of the signals is envisaged.

10.2 Computers as system components: devices that can change their state only at discrete times

A system that can change its state only at discrete points in time is called a discrete time system. Among the many examples of discrete time systems in everyday life, the rates of exchange for foreign currencies charged by retail banks could be mentioned. Typically, these rates may be updated once every working day and stay constant otherwise.

Computers are the discrete time systems that interest us here; in particular, computers that perform the same calculation repeatedly. Such computers are used, usually in the form of programmable logic controllers (PLCs), as controllers within closed loop systems. It turns out, perhaps surprisingly, that the discrete time effects of a computer, when used as a controller, are sufficiently profound to require a whole new batch of design techniques – these are introduced in this chapter.

To get a feel for what is going on, let us look at a very simple control loop first not containing a computer (case A), and second, containing a computer (case B).

The control loop (case A) simply comprises an integrator with negative feedback (Figure 10.1). Everything is at rest and set at zero and then v is moved instantaneously from $v = 0$ to $v = 1$. Simple calculation will show that the system output y moves as shown (Figure 10.2).

In case B, a computer 'looks at' the signal e every 1.5 s, multiplies this signal by unity and puts this out to the integrator where it remains constant for 1.5 s.

Essentially, cases A, B differ only in the interposition of a discrete time device in case B (Figure 10.3). To work out the response, we note that over the first 1.5 s period, the input to the integrator is fixed at $v = 1$. Thus,

$$y(t)|_{t=1.5} = \int_0^{1.5} e(t)dt = \int_0^{1.5} dt = 1.5$$

$$e(t)|_{t=1.5} = v(t)|_{t=1.5} - y(t)|_{t=1.5} = 1 - 1.5 = -0.5$$

and

$$y(t)|_{t=3} = \int_{1.5}^{3} -0.5dt + 1.5 = 0.75$$

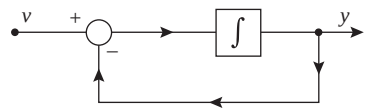


Figure 10.1 A continuous typical feedback loop with an integrator in the forward path (case A)

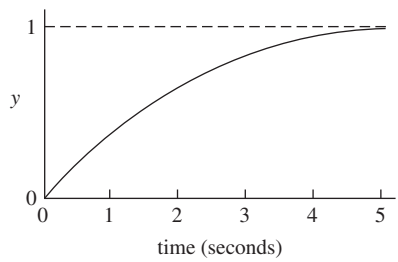


Figure 10.2 The step response of the system of Figure 10.1

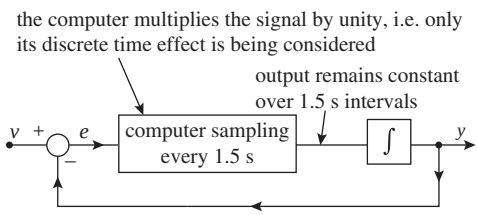


Figure 10.3 The system of Figure 10.1 with the addition of a computer that multiplies by unity and has a sampling interval of 1.5 s

and the response $y(t)$ is as shown in Figure 10.4. The significant differences between the responses in Figures 10.2 and 10.4 are due entirely to the effects of sampling.

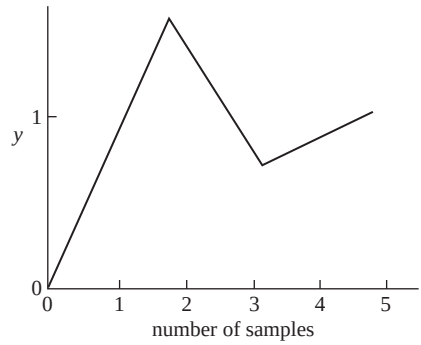


Figure 10.4 The step response of the system of Figure 10.3

10A A simple and informative laboratory experiment

It forms an interesting laboratory demonstration to reproduce the results of Figures 10.1–10.4 experimentally and then to vary the sampling interval of the computer, which is only a sample and hold device in reality, and observe the results. As the sampling interval is increased, instability will eventually occur. The demonstration can then be enhanced by connecting a frequency response analyser to determine approximately the phase shift characteristics of the computer as a function of applied frequency. A Bode plot check on stability will, very satisfyingly, be found to agree with experimental findings.

10.3 Discrete time algorithms

In this chapter, we are concerned with the discrete time control of continuous time processes (Figure 10.5). A discrete time algorithm is an algorithm that operates on a sequence of error signals to produce a sequence of command signals. The

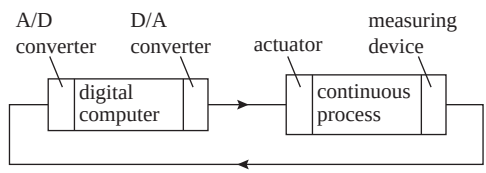


Figure 10.5 A continuous process under digital control

importance of discrete time algorithms lies in the fact that they are directly realisable in a digital computer controller. Such a digital controller samples the error at regular intervals of T seconds and produces a sequence of output commands, spaced at the same interval.

A continuous signal $e(t)$, when sampled every T seconds, is denoted e^* and the command sequence produced by a discrete time controller is denoted u^*. The discrete time command signal u^* must be converted into an analogue signal before being applied to a continuous process. Exact reconstruction of a continuous signal from samples is impossible to perform in real time since the reconstruction algorithm necessarily calls for unavailable future samples of the measured variable. Approximately correct reconstruction is possible but the necessary algorithms are relatively complex and they have undesirable frequency-response characteristics. Usual practice for conversion of the command sequence u^* into a continuous signal is a very crude piece-wise constant approximation. The device that performs such reconstruction is a digital to analogue converter whose input is updated every T seconds. Seen as a mathematical component, rather than as a physical device, the operation of piece-wise constant reconstruction is equivalent to that of a zero-order hold device.

10.4 Approaches to algorithm design

Roughly, there are two approaches to algorithm design.

10.4.1 Direct controller synthesis

Procedure in outline:

(i) Convert the specification that the final system must meet into a desired transfer function $H(z)$. This step will very often involve a considerable amount of approximation – particularly in those frequently encountered cases where the original specification is expressed in terms far removed from those pertaining to transfer functions.

However, if the specification can be expressed in terms of a desired natural frequency and a desired damping factor, then Figure 10.6 may be used directly to choose the poles of $H(z)$.

To use Figure 10.6, decide upon the required natural frequency ω_n, damping factor ζ, sampling interval T, and use the diagram to locate the intersection in the complex plane of the ω_n and the ζ loci. Suppose this intersection is at $a + jb$, then the poles of the sought-for transfer function $H(z)$ have to be located at $a + jb$. That is, the denominator of $H(z)$ should be $(z - a + jb)\,(z - a - jb)$.

Choice of the numerator of $H(z)$: In choosing the numerator of $H(z)$, the following factors need to be considered:

(a) Steady state response
(b) Frequency response
(c) Physical reachability and computational time requirements for the controller $D(z)$

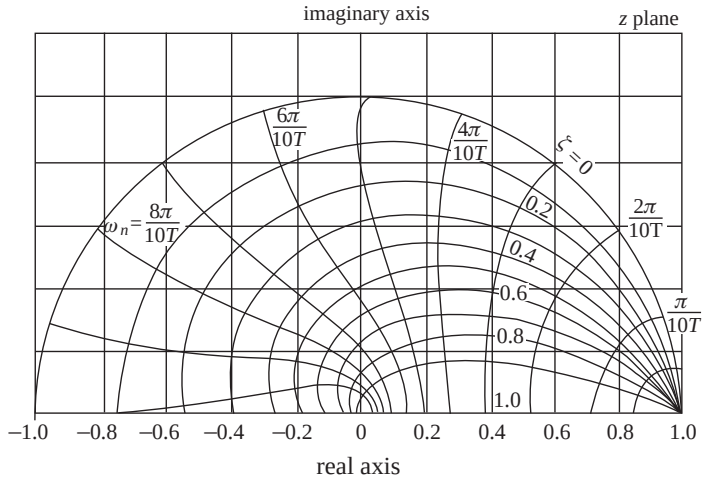


Figure 10.6 Diagram to assist in choosing the poles of H(z)

Considering (a), recall that the steady response to a unit step, for stable H, is $H(z)$ as $z \to 1$. Considering (b), one point of view is that the response of $H(z)$ when $\omega s = \omega s/2$ should be zero. Such behaviour can be obtained by placing one or more zeros at $z = 1$. Considering (c), notice that if the order of numerator and denominator of $D(z)$ are equal, then 'instantaneous' calculation of control outputs is implied. Choosing the order of numerator in H to be one less than the order of the denominator allows one sampling period T for the control algorithm to be calculated.

(ii) Produce a transfer function $G(s)$ representing the process that is to be controlled.
(iii) Form the transfer function $G'(s) = G_0(s)G(s)$, where G_0 is a model of the interface between controller and process.
(iv) Discretise the transfer function $G'(s)$ to produce the discrete time equivalent $G'(z)$.
(v) Use the relation $D(z) = H(z)/\{G'(z)[1 - H(z)]\}$ to synthesise the necessary controller for insertion into the loop (Figure 10.7).
(vi) Convert $D(z)$ into a difference equation and use it as a real-time algorithm.

Figure 10.7 The combination of controller D(z) and process + zero-order hold G′(z), in closed loop

10B A clever manipulation: how the digital to analogue convertor (zero-order hold) is transferred for calculation purposes to become part of the process to be controlled

(i) Notice carefully that in the approach described above, the digital to analogue convertor at the output of the controlling computer is grafted on to the process to form the artificial process G', made up as $G'(s) = G_0(s)G(s)$.

The design procedure is thus to control G' rather than G. Thus, insofar as there are distortions caused in the analogue signal reconstruction at the digital to analogue convertor, they, being embodied in G', will automatically be compensated during control algorithm design.

(ii) Notice also that

$$Z\{G_0(s)G(s)\} \neq Z\{G_0(s)\}Z\{G(s)\}$$

In fact,

$$Z\{G_0(s)\} = Z\left(\frac{1-\exp(-sT)}{s}\right) = \frac{(1-z^{-1})z}{(z-1)} = 1$$

i.e. a zero-order hold unconnected to another analogue device is invisible to the Z transform.

Comment: It can be seen that the equation for $D(z)$ contains models both of the process and the desired behaviour. In effect, the controller cancels out the existing process characteristics and replaces them by those of the required system.

10.4.2 Gain plus compensation approach

Idea in outline:

(i) If a controller consisting of only a simple gain of numerical value C is used as in Figure 10.8, then the performance of the resulting system (of transfer function $CG(z)/[1 + CG(z)]$) may be manipulated by choice of the value for C.

(ii) As C is increased, the speed of response of the system increases but, in general, the response becomes oscillatory, and as C is increased further, the system becomes unstable.

(iii) By incorporating a suitable compensator M into the loop (Figure 10.9), improved stability characteristics can be given to the loop and then the value of C can be further increased with a consequent increase in speed of response. This process of juggling the design of compensator M and the value of gain C can be iterated until a best possible response is achieved.

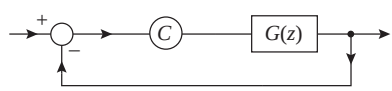


Figure 10.8 A controller consisting of a simple gain C *in a discrete time loop*

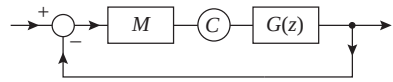


Figure 10.9 Incorporation of a compensator into the loop of Figure 10.8

The compensator M primarily needs to improve the stability margin of the loop, hence allowing higher gain C to be used, resulting in faster response. M may be an approximate differentiator, as in the three-term controller (the three parallel terms are again C, a differentiator D and an integrator I that is present to remove steady state error).

Three-term controllers are favoured by practitioners on the following grounds: one form of controller satisfies all applications; the controller is easily 'tuned' for application using Ziegler–Nichols rules (see Section 9.3.1); the controller is a successful workhorse being applied in huge numbers across industry.

Seen from a frequency response point of view, the compensator M is a phase-advance network and frequency response techniques, usually used in the s domain, allow the design to be matched to the application.

(iv) Discretise the MC combination to be directly implementable in a digital computer.

10C Takahashi's algorithm (The algorithm described here is interesting from a tutorial viewpoint, but see, for instance, Johnson (2005) for a treatment of more industrially applicable algorithms.)

In representing a typical process by discrete data points (assuming that a constant value of sampling interval T is to be used), in order to capture the all important initial curvature, a rather short value of T is indicated. However, in order to capture the (also important) final value, a large value of T is indicated – so that the number of points to be logged will not be excessive.

Takahashi solves this problem nicely by taking frequent samples initially in the step response and then using a formula to generate further points until the correct steady state is reached (Figure 10.10). Notice that these generated further points will not, in general, lie exactly on the curve.

Takahashi's algorithm then uses the model coefficients to synthesise a controller for the process (the one that generated the open loop step response) as follows (Figure 10.11): The model of form

$$G(z) = \sum_{i=1}^{n-1} g_i z^{-1} + \frac{g_n z^{-n}}{1 - pz^{-1}}$$

is fitted to the first n data points and the parameter p is fixed to give the correct steady state value and approximately correct decay rate. Takahashi

then derived formulae (Takahashi *et al.*, 1970) by which the $n + 1$ coefficients in the controller (Figure 10.11) may be calculated directly from the $n + 1$ model coefficients ($g_1, \ldots, g_n, p$).

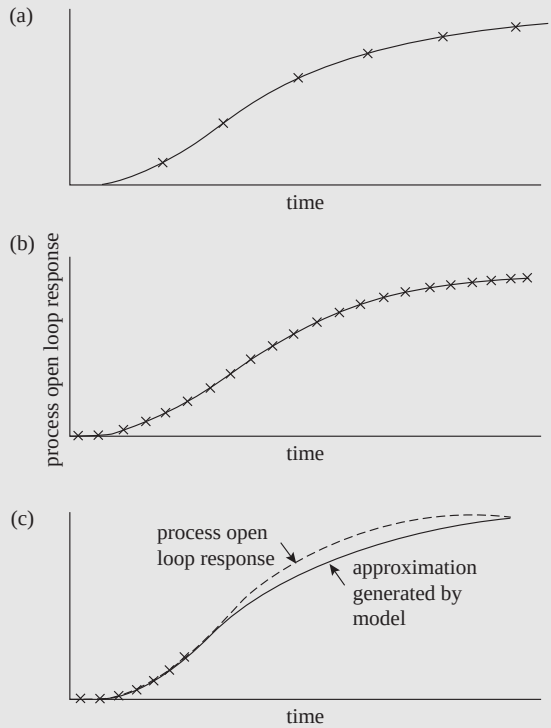


Figure 10.10 How many points are needed to capture a step response? (a) Too few points fail to capture the essential shape; (b) too many points to handle (bearing in mind that the order of the online algorithm will be the same as the number of points); (c) Takahashi's approach. Early points capture the essential shape. Approximation (shown dotted) completes the response

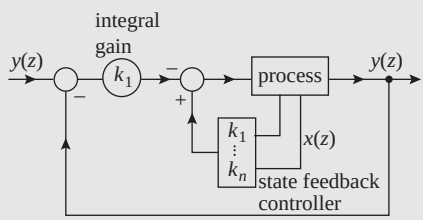


Figure 10.11 Takahashi's algorithm

10.5 Overview: concluding comments, guidelines for algorithm choice and some comments on procedure

(i) Very broadly, there are two approaches to algorithm design. The first, synthesis of $D(z)$ to achieve a specific closed loop transfer function $H(z)$, is theoretically sound but suffers from two defects: choosing $H(z)$ usually involves massive approximation; $D(z)$ 'contains' both $G(z)$ and $H(z)$ and is therefore often unwieldy. The second approach, using a gain plus compensator, is not very scientific but it has the great merit of simplicity.

(ii) Every continuous time algorithm can be discretised – this is one source of algorithms. Note, however, that the performance of a discretised algorithm is always degraded to some extent compared with that of the original continuous time algorithm. The extent of degradation is governed by the choice of sampling interval.

These are, however, discrete time algorithms that have no (apparent) continuous time equivalents. These are the most interesting algorithms and they tend to be incorporated as part of advanced control packages for solution of demanding problems.

(iii) Some industries, like aerospace, tend predominantly to use frequency response continuous time design methods and only later to discretise. Process industries tend to use off-the-shelf three-term algorithms integrated within diagnostic and monitoring supervisory software.

(iv) In general, it is recommended to use simple solutions (for instance, off-the-shelf three-term controllers) for routine problems. However, it is important to match the level of sophistication of the controller to the inherent difficulty of the problem.

(v) Many alternative methods have been put forward for the selection of sampling interval T. The one suggested here, based on closed loop bandwidth, is a reasonable compromise between ad hoc methods and theoretical overkill.

10D Some difficulties in moving from differential equations to approximating difference equations

Suppose that we have a differential equation

$$y''' + 3y'' + 2y' + y = 0 \qquad (10.1)$$
$$y(0) = 10,\ \ y'(0) = 2,\ \ y''(0) = 5$$

Suppose also that we have discretised the differential equation, by any suitable method, into the form

$$y(k) = ay(k-1) + by(k-2) + cy(k-3)$$

for some chosen time interval T and with numerical values being found for a, b, c.

Suppose, finally, that we wish to use the difference equation to generate an approximate numerical solution for the differential equation that it approximates. The differential equation has three initial conditions and the difference equation needs three starting values. However, it is not clear how to proceed or at least how to get started.

10E Discretisation

By discretisation, we mean the move from continuous to discrete time; differential equation to difference equation; s domain to z domain.

The most obvious approach to discretisation might appear to be replacement of s by its equivalent function in z. However, $z = \exp(st)$; hence, the required substitution would be $s = (\ln z)/T$. Substitution would then produce an ugly polynomial in $\ln z$.

Discretisation methods that are actually used are as follows:

(i) Replacing derivatives dy/dt by their finite difference approximations

$$\frac{y_{k+1} - y_k}{T}, \frac{y_k - y_{k-1}}{T}, \frac{((y_{k+1} + y_k)/T) - ((y_k - y_{k-1})/T)}{T}$$

(ii) Mapping the poles of a continuous transfer function $G(s)$ to the correct equivalent points in the z plane as dictated by the definition of z.

(iii) Using the relation

$$G(z) = Z\{\mathcal{L}^{-1}(G(s))\}$$

(iv) Converting $G(s)$ into multivariable $\{A, B, C\}$ form and using $\Phi(T)$, $\Psi(T)$ as discrete operators (see Chapter 11 for more background).

(v) Using any numerical algorithm for the time solution of differential equations, e.g. Runge–Kutta methods.

Discretisation needs care since it is easily possible for a stable $G(s)$ to be transformed into a $G(z)$ of quite different, even unstable, character.

10F A simple matter of quadratic behaviour

We investigate the problem: Given that $G(z)$ has the form

$$\frac{z - a}{(z - b)(z - 1)}$$

determine from first principles in the z plane the maximum value of C that does not produce instability in the loop (Figure 10.12).

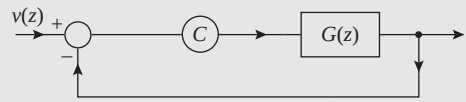


Figure 10.12 The closed loop system whose poles are studied in this section

Approach: The loop has the z transform

$$\frac{C(G(z))}{1 + CG(z)1 + CG(z)} = \frac{C(z-a)}{(z-b)(z-1) + C(z-a)}$$

We seek the largest value of C for which the roots of $1+CG(z) = 0$ satisfy $|z| < 1$.

Now, from an examination of the equation, we can see that as $C \to \infty$, the two solutions will have asymptotes $z \to \infty$, $z \to a$.

It could seem to the uninitiated that the value of C we are seeking might be the value of C that brings one root of the equation to $z = -1$.

Question: When will the simple stability test

$$1 + CG(z)|_{z=-1} = 0 \tag{10.2}$$

yield the required value of C?

Test cases (Figure 10.13)

(1) $$G(z) = \frac{z + 0.2}{(z - 0.3)(z - 1)}$$

(2) $$G(z) = \frac{z + 0.2}{(z - 0.4)(z - 1)}$$

The point to note from these diagrams is that in Figure 10.13(a) the root locus leaves the unit circle at $z = -1$, whereas in Figure 10.13(b), the locus enters the circle at that point – numerical checks on stability can be misleading unless the locus is drawn.

Applying the simple test (10.2) to the two cases leads, respectively, to the solutions:

(i) $z^2 - 1.3z + 0.3 + Cz + 0.2C|_{Z=1} = 0 \Rightarrow C = 3.25$
(ii) $z^2 - 1.4z + 0.4 + Cz + 0.2C|_{Z=1} = 0 \Rightarrow C = 3.5$

Case 1 with $C = 3.25$ leads to roots at $z = -0.95$, $z = -1$
Case 2 with $C = 3.5$ leads to roots at $z = -1$, $z = -1.1$

i.e. for case 1, $C_m = 3.25$ is confirmed as correct, but for case 2, we find that $C_m < 3.5$.

To investigate, we plot the loci of the roots of (10.12) as C varies. It is now clear that the difficulty in case 2 arises because the loci leave the unit

circle at points where z has complex values. Calculation shows that this behaviour occurs whenever

$$a \leq -\left(\frac{1-b}{3+b}\right) \tag{10.3}$$

and that the value C_m of C at which the loci leave the unit circle is then

$$C_m = \frac{1-b}{|a|} \tag{10.4}$$

Using this equation we obtain the correct value of C_m for case 2 as $C_m = 3.0$. Of course, when the inequality (10.3) is not satisfied, C_m can be determined using (10.4).

Using only a knowledge of elementary quadratic equations, we have obtained an interesting insight into the behaviour of a closed loop discrete time system.

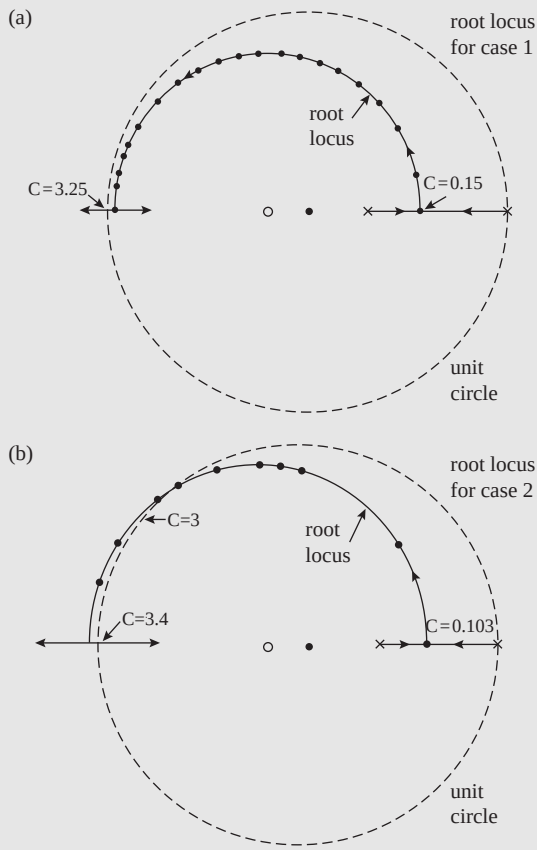


Figure 10.13 Root loci (upper halves only shown) for the system of Figure 10.12: (a) with G(z) = (z + 0.2)/(z − 0.3)(z − 1)*; (b) with* G(z) = (z + 0.2)/(z − 0.4)(z − 1)

10G Continuous is not the limit of discrete as T→0

Consider the transfer function

$$G(s) = \frac{1}{s + 0.1}$$

The equivalent discrete time transform, obtained by taking the Z transform $\mathcal{L}^{-1}\{G(s)\}$ is

$$G(z) = \frac{z}{z - \exp(-0.1T)}$$

If we set T at some reasonable value, say $T = 1$, the behaviour of the inverse transform of $G(z)$ in the time domain approximates reasonably well the behaviour of the inverse transform of $G(s)$.

We might assume that as $T \to 0$, the approximation will improve until, in the limit, the two behaviours coincide. However, note that

$$G(z)|_{T\to 0} = \frac{z}{z - 1}$$

whose s domain equivalent is $1/s$, an integrator. (Attempts to investigate this effect by numerical methods tend to run into problems of word length.)

10H Non-uniqueness of inverse Z transforms

From the point of view of the Z transform, the three signals shown in Figure 10.14 are identical. This leads to many practical problems, since, if the signals are input to a system, the effect of the three signals will be markedly different. Similarly, a signal that is apparently constant, according to the transform, may actually be oscillating widely between sampling instants.

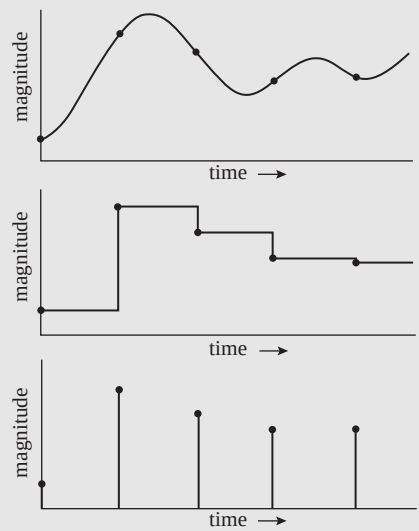


Figure 10.14 The three signals shown have identical Z transforms

10I The stability of a system can usually be considered independently of the nature of the inputs that it receives, but here is a counter-example

Stability is normally considered to be an inherent property of a system so that, for any bounded input, a stable system should produce a bounded output.

However, note the following: A system of transfer function

$$G(z) = \frac{1}{z^2 + 1}$$

in response to a step $u(k) = \{1, 1, 1, \dots\}$ produces the bounded output

$$y(k) = \{0, 0, 1, 1, 0, 0, 1, 1, 0, 0, \dots\}$$

but in response to the input

$$u(k) = \{1, 0, -1, 0, 1, 0, -1, \dots\}$$

it produces the unbounded output

$$y(k) = \{0, 0, 1, 0, -2, 0, 3, 0, -4, 0, 5, 0, -6, 0, \dots\}$$

(Further investigation will show that the input for the second case has $u(z) = 1/(z^2+1)$ so that $G(z)u(z)$ has replaced poles on the unit circle.)

Note: Source material and suggestions for further reading to support the topics of this chapter will be found in Chapter 18.

Chapter 11

Multivariable linear systems and the state space approach

11.1 Introduction

The classical control approaches described in earlier chapters were largely developed in the period 1930–1960. Those approaches, mostly based around frequency response foundations, were, in general, easily understood and directly industry applicable. Since 1960, when, not coincidentally, the first World Congress of IFAC (the International Federation of Automatic Control) was held in Moscow, the mainstream efforts of control systems researchers have increasingly centred around multivariable systems described by state space representations.

Universally available computer software now makes most of the manipulations required in control systems analysis and design a routine matter. However, it is still essential to look deeper than mere manipulation, and for the state space approach, acquiring a good intuitive understanding of what is happening takes quite an effort. This chapter concentrates on establishing the mainstream structure of the state space approach and Chapter 12 will assist that development of understanding by linking back to the earlier chapters on classical control.

The biggest change for the reader, coming to state space methods for the first time, is possibly that state space representations abandon the input–output viewpoint of classical control in favour of an input-state, state-output viewpoint. The state vector, always denoted x, is absolutely central to the state space viewpoint.

11.1.1 State space representations

In the *state space* modelling of linear systems, it is assumed that there exists an nth order vector called the state vector, whose value at every instant of time completely characterises the dynamic state of the system. The order n is, in general, equal to the sum of the orders of all the individual differential equations that together describe the system.

11.1.2 The state vector

Imagine the ongoing time solution of an nth order linear differential equation and track the progress of the n initial conditions. This progress can be visualised as the trajectory of a point in n-dimensional linear space – the state space. At any time t,

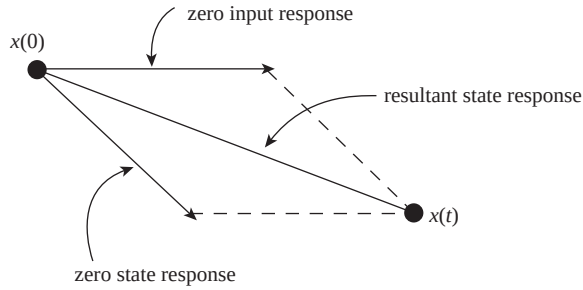


Figure 11.1 Illustrating how the trajectory of the state through state space is the vector sum of the response to initial conditions and the response to forcing inputs

that trajectory can be completely specified by an n-dimensional vector: the state vector. To qualify for the name, state vector – it must completely describe the dynamic state of the system of that time (with 'completely' being defined by axioms that will be encountered shortly).

Linearity allows the state response trajectory to be very usefully decomposed into two parts (i) that part of the response due only to initial conditions $x(0)$ and (ii) that part of the response due only to external inputs $u(\tau)$, $\tau \in [0, t]$ as indicated in Figure 11.1.

Every nth-order single-input, single-output linear system can be decomposed into n first-order equations and described in state space form. We choose such a system to illustrate some simple state space ideas. Let the single-input, single-output process be

$$\frac{d^3y}{dt^3} + \frac{2d^2y}{dt^2} + \frac{3dy}{dt} + 4y = u \tag{11.1}$$

To move to a state space model we let

$$x_1 = y$$
$$x_2 = \dot{x_1}$$
$$x_3 = \dot{x_2}$$

Then, equivalent to (11.1), we can write

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -4x_1 - 3x_2 - 2x_3 + u$$

This is the state space form. It would more usually be written

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u$$

$$y = (1 \quad 0 \quad 0) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

which is usually written

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}\right\} \tag{11.2}$$

And this formulation is the same for all multivariable linear systems. (Although for completeness it should be noted that the last equation is sometimes augmented to become

$$y = Cx + Du$$

to allow for those (rare) situations where the system output has direct non-dynamic links to the system input.)

The remainder of the chapter establishes state-variable techniques for the representation and analysis of both continuous time and discrete time systems with an analogous development for the two cases. Canonical forms are introduced for the structural insight that they create and it is also indicated how canonical forms may be useful in control system design.

11.2 The concept of state

Consider a mathematical model that consists of n linear time-invariant, first-order ordinary differential equations:

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}x_j + \sum_{j=1}^{r} b_{ij}u_j \tag{11.3}$$

Define the vectors

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix}$$

If we are given $x(t_0)$ and $u(\tau)$ for all $\tau \geq t_0$, then we can determine $x(t)$ for any $t > t_0$. We do not require information about $x(t)$ for $t < t_0$ since all necessary information is assumed to be contained in the vector $x(t_0)$. A vector having this property is called a *state vector*. It is often useful to consider the state vector as an element in an appropriate linear space, which is then defined as the *state space* for the system.

For a particular system Σ, the state space is not uniquely defined and many possible descriptions exist. However, the minimum possible dimension for the state space is fixed for any particular system Σ. (In the above example, no vector of dimension less than n can have the required properties needed by a state vector.)

11.3 Alternative system descriptions

A linear continuous multivariable process is shown in Figure 11.2. The r input variables $u_1, \ldots, u_r$ can be represented as a vector u in an r-dimensional linear space U. Similarly, the state vector x is an element in the state space X and the output vector y is an element in the output space Y. Table 11.1 summarises this information.

The process will be referred to by the symbol Σ. Assume that the process Σ is time invariant, then it can be represented mathematically in three ways:

(a) By two linear mappings (we shall designate these ϕ and η) linking the three spaces U, X, Y. The process Σ can then be represented in *mapping form*

$$\Sigma = \{I, U, X, Y, \phi, \eta\} \tag{11.4}$$

where I is the space representing time.

(b) By a set of n ordinary differential equations of the form

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_j + \sum_{j=1}^{r} b_{ij} u_j, \quad i = 1, \ldots, n \tag{11.5}$$

and a set of m algebraic equations of the form

$$y_i = \sum_{j=1}^{m} c_{ij} x_j, \quad i = 1, \ldots, m \tag{11.6}$$

The equations can be collected into *vector–matrix form*

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}\right\} \tag{11.7}$$

The vector x is required to have the state property.

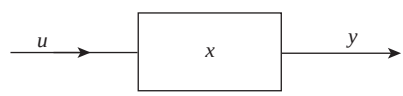


Figure 11.2 A general model for a multivariable system

Table 11.1 The usually accepted designations for input, state and output spaces

	Vector			**Space**
u	r dimensional	Input vector	U	Input space
x	n dimensional	State vector	X	State space
y	m dimensional	Output vector	Y	Output space

(c) By a set of $r \times m$ transfer functions, $g_{ij}(s)$, linking each of the inputs with each of the outputs. Define the *transfer matrix* $G(s)$ as the matrix whose elements are the $g_{ij}(s)$, then to represent Σ we can write

$$y(s) = G(s)u(s) \tag{11.8}$$

where

$$y(s) = \begin{bmatrix} y_1(s) \\ \vdots \\ y_m(s) \end{bmatrix}, \quad u(s) = \begin{bmatrix} u_1(s) \\ \vdots \\ u_r(s) \end{bmatrix}$$

Equation (11.8) is the transfer matrix representation of the system.

11.4 The mapping representation of Σ

The mapping representation is valuable for developing insight into system fundamentals and the unification of the subject through simple broad theorems.

The description of the process Σ by the sextuplet $\{I, U, X, Y, \phi, \eta\}$ is completed by the following assumptions, or axioms.

(1) $\phi : X \times U \to X, \qquad \phi(t_0, u(t)), t_1, x(t_0)) = x(t_1)$

(2) $\eta : X \to Y, \quad \eta(t_0, x(t_0)) = y(t_0)$

The mappings ϕ, η, operating as shown, are linear mappings. For a continuous time process, I is replaced by the real line $\mathbb{R}^1$ representing the underlying continuous time set. (For a discrete time process, I is replaced by the set of integers, denoted Z.)

(3) $\phi(t_0, u(t), t_2, x(t_0)) = \phi(t_1, u(t), t_2, (\phi(t_0, u(t), t_1, x(t_0)))$

This, the semi-group property, simply ensures that the solution at time t_2 is the same whether calculated in one large time step $t_2 - t_0$, or in two smaller time steps $t_1 - t_0$ and $t_2 - t_1$

(4) $\phi(t_0, u(t), t_0, x(t_0)) = x(t_0)$

This property is clearly necessary.

(5) $\phi(t_0, u_1(t), t_1, x(t_0)) = \phi(t_0, u_2(t), t_1, x(t_0)), t_1 > t_0$

provided that

$$u_1(t) = u_2(t) \text{ on the interval } [t_0, t_1]$$

This interesting axiom ensures that the system is causal (i.e. it cannot respond to a stimulus before the stimulus is received) and hence physically realisable. It also ensures that x really does have the property necessary in a state vector, as follows:

Given a knowledge of the state vector $x(t_0)$ at time t_0 and a knowledge of the input on any time interval $[t_0, t]$, $t > t_0$, the state $x(t)$ can be determined.

The implication is that $x(t_0)$ contains sufficient information to make knowledge of events before time t_0 unnecessary, i.e. x is a state vector.

(6) U is a space of piecewise continuous functions on I.

One consequence of the axioms is that we can write

$$\phi(t_0, u(t), t_1, x(t_0)) = \phi(t_0, 0, t_1, x(t_0)) + \phi(t_0, u(t), t_1, 0).$$

The right-hand terms are called the zero-input response and the zero-state response, respectively.

Let X be a linear space of finite dimension n, then Σ is called a *linear finite-dimensional system*.

Let X be a linear system of infinite dimension, then Σ is called a *linear infinite-dimensional system.*

To appreciate the simplicity of the axioms, they should be visualised in a geometric setting. For instance, axiom 3 can be visualised with the help of Figure 11.3 and axiom 5 with the help of Figure 11.4.

The study of systems through their mappings is most rewarding.

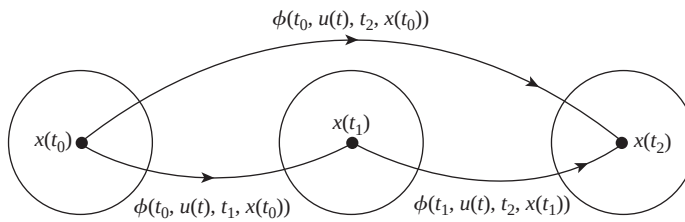


Figure 11.3 Visualisation of axiom 3

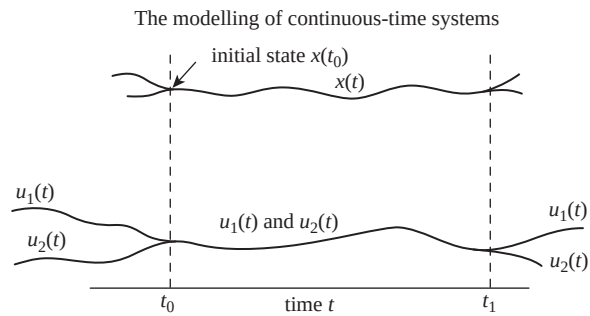


Figure 11.4 Visualisation of axiom 5

11.5 The modelling of continuous time systems through state space equations

11.5.1 A general non-linear model

A general model for a multivariable system was shown in Figure 11.2. The system receives r input signals denoted $u_1, \ldots, u_r$ and gives out m output signals, denoted $y_1, \ldots, y_m$. It is convenient to define the vectors

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

The *input vector* u carries information on the external influences (applied controls and disturbances) affecting the system. The *output vector* y represents the system measurements.

To complete the description, n internal states, $x_1, \ldots, x_n$ are assumed.

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is recognised as the *state vector* of the system. The state vector has been defined in Section 11.2.

The input, state and output vectors are considered to be members of the *input*, *state* and *output spaces*, respectively, denoted U, X and Y.

With this background, we are ready to represent the general multivariable system by the following set of equations:

$$\begin{aligned} \dot{x}_i &= f_i(x_1, \ldots, x_n, u_1, \ldots, u_r, t), \quad i = 1, \ldots, n \\ y_i &= g_i(x_1, \ldots, x_n, t), \quad i = 1, \ldots, m, \end{aligned} \tag{11.9}$$

in which the f_i and g_i are non-linear functions and t represents time. We assume that $t \in I$, where $I = \mathbb{R}^1$. The inter-relation of the functions and the spaces can be expressed by the relations.

$$\begin{aligned} f_i &: X \times U \times I \to X \\ g_i &: X \times I \to Y \end{aligned}$$

11.5.2 Linearisation of the model

Multivariable control theory rests on a foundation of algebra and operator theory and the model (11.9) derived above must be linearised before proceeding further.

Suppose that the functions f_i, g_i defined in (11.9) are in fact linear, then necessarily (as a consequence of the representation theorem of real analysis) the following relations hold:

$$\begin{aligned} \dot{x}_i &= a_{i1}(t)x_1 + \cdots + a_{in}(t)x_n + b_{i1}(t)u_1 + \cdots + b_{ir}(t)u_r, \quad i = 1, \ldots, n \\ y_i &= c_{i1}(t)x_1 + \cdots + c_{in}(t)x_n, \quad i = 1, \ldots, m \end{aligned} \tag{11.10}$$

Or in vector–matrix form

$$\begin{aligned} \dot{x} &= \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix} x + \begin{bmatrix} b_{11}(t) & \cdots & b_{1r}(t) \\ \vdots & & \vdots \\ b_{n1}(t) & \cdots & b_{nr}(t) \end{bmatrix} u \\ y &= \begin{bmatrix} c_{11}(t) \\ \vdots \\ c_{m1}(t) \end{bmatrix} \cdots \begin{bmatrix} c_{1n}(t) \\ \vdots \\ c_{mn}(t) \end{bmatrix} x \end{aligned} \tag{11.11}$$

or

$$\dot{x} = A(t)x + B(t)u, \quad y = C(t)x$$

Returning to the general case where the functions f, g are non-linear, then, provided that the functions are differentiable, local approximations to the matrices $A(t)$, $B(t)$ and $C(t)$ can be determined from the following relations (the operation is performed along nominal trajectories for x and u):

$$A(t) = \left(\frac{\partial f_i}{\partial x_j} \right) \Bigg|_{\substack{x \\ u}} \tag{11.12}$$

$$B(t) = \left(\frac{\partial f_i}{\partial u_j} \right) \Bigg|_{\substack{x \\ u}} \tag{11.13}$$

$$C(t) = \left(\frac{\partial g_i}{\partial x_j} \right) \Bigg|_{x} \tag{11.14}$$

11.6 Calculation of time solutions using the transition matrix

11.6.1 The time-invariant case

11.6.1.1 Definition of the matrix exponential in terms of an infinite series

First, we define the *matrix exponential* e^{AT} as an infinite series:

$$e^{AT} = I + AT + \frac{1}{2!}(AT)^2 + \frac{1}{3!}(AT)^3 + \cdots$$

the series being convergent for any finite T.

11.6.1.2 The transition matrix

Given the equation

$$\dot{x} = Ax + Bu \tag{11.15}$$

put $z = e^{-At}x$. Then (the reader should confirm the next step from first principles using the infinite series definition of e^{At}),

$$\begin{aligned}\dot{z} &= -Ae^{-At}x + e^{-At}\dot{x}\\ &= -Ae^{-At}x + e^{-At}(Ax + Bu)\\ &= e^{-At}Bu\end{aligned}$$

Integrate from t_0 to t

$$\begin{aligned}z(t) &= z(t_0) + \int_{t_0}^{t} e^{-At}Bu(\tau)d\tau\\ e^{-At}x &= e^{-At_0}x(t_0) + \int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau\\ x(t) &= e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau\end{aligned} \tag{11.16}$$

The *transition matrix* $\Phi(t - t_0)$ is now defined such that

$$x(t) = \Phi(t - t_0)x(t_0) + \int_{t_0}^{t} \Phi(t - \tau)Bu(\tau)\mathrm{d}\tau \tag{11.17}$$

Numerical solution can be achieved by choosing a time step T, such that over every T, $u(t)$ can be considered constant. Then (11.17) can be integrated to yield

$$x(T) = \Phi(T)x(0) + A^{-1}(\Phi(T) - I)Bu(0)$$

and, in general,

$$x((k+1)T) = \Phi(T)x(kT) + \Psi(T)u(kT) \tag{11.18}$$

where $\Psi(T)$ is defined by the relation

$$\Psi(T) = A^{-1}(\Phi(T) - I)B$$

11.6.1.3 Laplace transform approach

Consider the equation

$$\dot{x} = Ax$$

in which A is an $n \times n$ matrix. Laplace transformation gives

$$\begin{aligned} sx(s) - x(0) &= Ax(s) \\ (sI - A)x(s) &= x(0) \\ x(t) &= \mathcal{L}^{-1}\{(sI - A)^{-1}\}x(0) \end{aligned}$$

from which, by uniqueness,

$$\Phi(t) = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$$

11.6.1.4 Diagonalisation approach

Let λ_i, e_i, $i = 1, \ldots, n$ be the (distinct) eigenvalues and eigenvectors, respectively, of the system matrix A. Define the *modal matrix* E by the relation

$$E = (e_1, e_2, \ldots, e_n)$$

and define $v = E^{-1}x$. Then,

$$E\dot{v} = AEv$$

$$\dot{v} = E^{-1}AEv = E^{-1}(AE)v = E^{-1}\left(E\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}\right)v = \Lambda v$$

where Λ is a diagonal matrix

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

of eigenvalues. Because of the diagonality of Λ, the transition matrix is particularly simple and we can write

$$v(t) = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ \vdots & \ddots & \vdots \\ 0 & & e^{\lambda_n t} \end{bmatrix} v(0) \triangleq e^{\Lambda t}v(0)$$

Should we wish to return to the original system co-ordinates, we have to make the substitution $v = E^{-1}x$ to obtain

$$x(t) = E\, e^{\Lambda t} E^{-1} x(0) = E\, e^{(E^{-1}AE)t} E^{-1} x(0)$$

from which

$$\Phi(t) = E\, e^{\Lambda t} E^{-1}$$

A variety of other methods is available for the calculation of the transition matrix but these three methods (infinite series, Laplace method, diagonalisation method) will cover all our needs. Commenting briefly on the comparative merits of the three methods we note:

(a) The Laplace method is always applicable but its use entails the inverse Laplace transformation of n^2 terms that becomes unwieldy for large systems. The method is relatively difficult to mechanise because of the inverse transformation step.
(b) The diagonalisation method only works unmodified for systems with distinct eigenvalues. It is relatively easy to mechanise.
(c) The series summation method is simple to mechanise although scaling and convergence problems may arise. For large t, very many terms have to be taken before convergence is obtained. The result from the series method is a matrix of numbers rather than of mathematical functions and this may limit the usefulness of the method for some applications.

Example Compute the transition matrix $\Phi(t)$ for the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}$$

by the three methods given above.

Laplace method

$$sI - A = \begin{bmatrix} s & -1 \\ 4 & s+5 \end{bmatrix}, \quad (sI - A)^{-1} = \frac{1}{s(s+5)+4} \begin{bmatrix} s+5 & 1 \\ -4 & s \end{bmatrix}$$

$$\Phi(t) = \mathscr{L}^{-1} \left\{ \begin{matrix} \dfrac{s+5}{s^2+5s+4} & \dfrac{1}{s^2+5s+4} \\ \dfrac{-4}{s^2+5s+4} & \dfrac{s}{s^2+5s+4} \end{matrix} \right\}$$

$$= \frac{1}{3} \mathscr{L}^{-1} \begin{matrix} \left(\dfrac{4}{s+1} - \dfrac{1}{s+4}\right) & \left(\dfrac{1}{s+1} - \dfrac{1}{s+4}\right) \\ \left(\dfrac{-4}{s+1} + \dfrac{4}{s+4}\right) & \left(\dfrac{-1}{s+1} + \dfrac{4}{s+4}\right) \end{matrix}$$

$$= \frac{1}{3} \left(e^{-t} \begin{bmatrix} 4 & 1 \\ -4 & -1 \end{bmatrix} - e^{-4t} \begin{bmatrix} 1 & 1 \\ -4 & -4 \end{bmatrix} \right)$$

Diagonalisation method Eigenvalues are found from the characteristic equation $\lambda^2 + 5\lambda + 4 = 0$ yielding $\lambda_1 = -4$, $\lambda_2 = -1$. Eigenvectors are

$$\begin{bmatrix} 1 \\ -4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The modal matrix

$$E = \begin{bmatrix} 1 & 1 \\ -4 & -1 \end{bmatrix}, \quad E^{-1} = \frac{1}{3}\begin{bmatrix} -1 & -1 \\ 4 & 1 \end{bmatrix}$$

$$\Phi(t) = Ee^{\Lambda t}E^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -4 & -1 \end{bmatrix}\begin{bmatrix} e^{-4t} & 0 \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 4 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\left(e^{-t}\begin{bmatrix} 4 & 1 \\ -4 & -1 \end{bmatrix} - e^{-4t}\begin{bmatrix} 1 & 1 \\ -4 & -4 \end{bmatrix}\right)$$

Series summation method

(a) With $t = 0.1$,

$$\Phi(0.1) = I + A(0.1) + \frac{A^2(0.1)^2}{2!} + \cdots$$

Let S_R be the summation after R terms, then evaluating numerically we obtain the sequence in Table 11.2.

(b) With $T = 1$,

$$\Phi(1) = I + A + \frac{A^2}{2!} + \cdots$$

we obtain the results shown in Table 11.3.

Table 11.2 Steps in the convergence of the series summation method when calculating the transition matrix using $T = 0.1$ second

R	S_R	R	S_R
1	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	5	$\begin{bmatrix} 0.983 & 0.0781 \\ -0.313 & 0.592 \end{bmatrix}$
2	$\begin{bmatrix} 1 & 0.1 \\ -0.4 & 0.5 \end{bmatrix}$	6	$\begin{bmatrix} 0.9830 & 0.7817 \\ -0.3127 & 0.5921 \end{bmatrix}$
3	$\begin{bmatrix} 0.98 & 0.075 \\ -0.3 & 0.605 \end{bmatrix}$	7	$\begin{bmatrix} 0.9830 & 0.7817 \\ -0.3127 & 0.5921 \end{bmatrix}$
4	$\begin{bmatrix} 0.983 & 0.0785 \\ -0.314 & 0.59 \end{bmatrix}$		

Table 11.3 Steps in the convergence of the series summation method when calculating the transition matrix using $T = 1$ second

R	S_R	R	S_R
1	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	11	$\begin{bmatrix} 0.46 & 0.09 \\ -0.36 & 0.006 \end{bmatrix}$
2	$\begin{bmatrix} 1 & 1 \\ -4 & -4 \end{bmatrix}$	12	$\begin{bmatrix} 0.49 & 0.125 \\ -0.5 & -0.13 \end{bmatrix}$
3	$\begin{bmatrix} -1 & -1.5 \\ 6 & 6.5 \end{bmatrix}$	13	$\begin{bmatrix} 0.48 & 0.11 \\ -0.45 & -0.09 \end{bmatrix}$
4	$\begin{bmatrix} 2.3 & 2 \\ -8 & -7.6 \end{bmatrix}$	14	$\begin{bmatrix} 0.48 & 0.12 \\ -0.47 & -0.10 \end{bmatrix}$
5	$\begin{bmatrix} -1.17 & -1.5 \\ 6.16 & 6.5 \end{bmatrix}$	15	$\begin{bmatrix} 0.48 & 0.116 \\ -0.46 & -0.097 \end{bmatrix}$
6	$\begin{bmatrix} 1.6 & 1.3 \\ -5.2 & -4.8 \end{bmatrix}$	16	$\begin{bmatrix} 0.484 & 0.116 \\ -0.466 & -0.098 \end{bmatrix}$
7	$\begin{bmatrix} -0.22 & -0.6 \\ 2.38 & 2.78 \end{bmatrix}$	17	$\begin{bmatrix} 0.4844 & 0.1165 \\ -0.4660 & -0.0982 \end{bmatrix}$
8	$\begin{bmatrix} 0.85 & 0.49 \\ -1.95 & -1.58 \end{bmatrix}$	18	$\begin{bmatrix} 0.4844 & 0.1165 \\ -0.4661 & -0.09821 \end{bmatrix}$
9	$\begin{bmatrix} 0.31 & -0.05 \\ 0.21 & 0.58 \end{bmatrix}$	19	$\begin{bmatrix} 0.4844 & 0.1165 \\ -0.4661 & -0.09821 \end{bmatrix}$
10	$\begin{bmatrix} 0.55 & 0.18 \\ -0.74 & -0.37 \end{bmatrix}$		

The disadvantage of poor convergence of this method is seen here.

11.6.2 The time-varying case

Consider the equation

$$\dot{x}(t) = A(t)x(t), \quad x(t) = x_0 \tag{11.19}$$

where $x \in \mathbb{R}^n$ and A is a square matrix of absolutely integrable functions defined on (t_0, t_1); i.e.

$$a_{ij}(t) \in L_1(t_0, t_1) \quad \text{for} \quad i,j \leq n$$

Notice that the elements of $A(t)$ are not required to be continuous for the system to be well posed.

In fact, if $||A(t)|| < g(t)$, $\forall t$, while $\int_{t_0}^{t_1} g(t)dt < \infty$, then (11.19) has a unique solution over (t_0, t_1) that is continuous in t.

The solution of (11.19) can still be written $x(t) = \Phi(t, t_0)x_0$, where Φ is the transition matrix associated with $A(t)$.

However, the transition matrix is more difficult to determine than for the time-invariant case, and before discussing this aspect we need more theoretical background concerning the transition matrix.

First, we define the *fundamental matrix.*

Given an equation

$$\dot{x}(t) = A(t)x(t) \tag{11.20}$$

consider the associated equation

$$\dot{X} = A(t)X \tag{11.21}$$

where X is an $n \times n$ matrix.

Given $X(t_0) = X_0$, with $\det X_0 \neq 0$, (11.21) can be solved to yield a solution $X(t)$, and $X(t)$ is called the *fundamental matrix* associated with (11.21). It satisfies the condition

$$\det X(t) \neq 0, \quad \forall t$$

Now let $X(t_0) = I$, then the solution $X(t)$ of (11.21) is called the *transition matrix* of $A(t)$. (This is the general definition of the transition matrix Φ as opposed to that given in (11.17), which applies only to the time-invariant case. Note carefully that in the time-invariant case, Φ is a function of an interval, whereas here, Φ is a function of initial and final times.) The transition matrix has the important characterising property

$$\begin{aligned} &\frac{\partial}{\partial t}(\Phi(t, t_0)) = A(t)\Phi(t, t_0), \quad \forall\ t \geq t_0 \\ &\Phi(t_0, t_0) = I \end{aligned} \tag{11.22}$$

The transition matrix transforms the initial condition $x(t_0)$ into the state $x(t)$ and gives the complete solution of the autonomous equation (11.19). To show this, differentiate the solution

$$x(t) = \Phi(t, t_0)x(t_0)$$

yielding

$$\dot{x}(t) = \dot{\Phi}(t, t_0)x(t_0)$$

using (11.22)

$$\dot{x}(t) = A(t)\Phi(t, t_0)x(t_0) = A(t)x(t)$$

which is the same as (11.19).

As we have seen

$$\left.\begin{aligned} &\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0) \\ &\Phi(t_0, t) = I \end{aligned}\right\} \tag{11.23}$$

has the solution $\Phi(t, t_0)$, which is the transition matrix we require.

The solution of (11.23) is given by the *Peano–Baker* series.

$$\Phi(t,t_0) = I + \int_{t_0}^{t} A(\tau_1)\mathrm{d}\tau_1 + \int_{t_0}^{t} A(\tau_1)\int_{t_0}^{\tau_1} A(\tau_2)\mathrm{d}\tau_2\mathrm{d}\tau_1 + \cdots \tag{11.24}$$

Theorem If A is a matrix of constant coefficients, then

$$\Phi(t,t_0) = I + A(t-t_0) + \frac{A^2(t-t_0)^2}{2!} + \cdots \tag{11.25}$$

Proof Since A is constant we can write

$$\Phi(t,t_0) = I + A\int_{t_0}^{t} \mathrm{d}\tau_1 + \mathrm{A}^2\int_{t_0}^{t}\int_{\tau_0}^{\tau_1} \mathrm{d}\tau_2\mathrm{d}\tau_1 + \cdots \tag{11.26}$$

but it is important to note that, in general,

$$\Phi(t,t_0) \neq \exp\left(\int_{t_0}^{t} A(\tau)\mathrm{d}\tau\right) \tag{11.27}$$

However, the equality does hold if

$$A(t)\int_{t_0}^{t} A(\tau)\mathrm{d}\tau = \int_{t_0}^{t} A(\tau)\mathrm{d}\tau A(t) \tag{11.28}$$

i.e. if the matrices $(A(t), \int_{t_0}^{t} A(\tau)d\tau)$ commute.

These matrices commute in case the elements of $A(t)$ are time invariant or if $A(t)$ is a diagonal matrix or in case $A(t)$ can be decomposed into a constant matrix M and a scalar $\alpha(t)$ so that $A(t) = \alpha(t)M$.

It is only in the case where the above matrices commute that $A(t)$ can be brought out of the integral during the derivation to yield

$$\frac{d}{dt}\left(\exp\left(\int_{t_0}^{t} A(\tau)d\tau\right)\right) = A(t)\exp\left(\int_{t_0}^{t} A(\tau)d\tau\right) \tag{11.29}$$

In the general case, the transition matrix is not the exponential of the integral of A.

Therefore, for linear time-varying problems, $\Phi(t, t_0)$ has to be determined by numerical evaluation of the expression (11.24). For simple cases, however, (11.24) can be expanded and solved analytically.

Not only is this much more time consuming than the evaluation of Φ for time-invariant systems, but also the same matrix cannot be used repetitively to advance the solution as in the time-invariant case.

If the system is time varying in a known deterministic manner, it may be possible to make a transformation such that the system appears invariant with respect to a new substituted variable. Cases where such transformation can be made are understandably rare.

Section 11.6.3 describes the periodically varying case, where such a transformation can be made with advantage.

11.6.3 The periodically time-varying case

In the modelling of rotating devices, such as radar antennae, periodically time-varying parameters are encountered.

Consider a model of the form

$$\dot{x}(t) = A(t)x(t), \quad A(t+T) = A(t) \tag{11.30}$$

where T is the period of variation.

The transition matrix $\Phi(t, t_0)$ is periodic in T. Define a matrix R by the relation

$$e^{RT} = \Phi(T, 0) \tag{11.31}$$

Such an R exists if $\Phi(T, 0)$ is non-singular, since Φ can then be diagonalised to give

$$\begin{bmatrix} \phi_{11} & & \\ & \ddots & \\ & & \phi_{nn} \end{bmatrix}$$

so that R could be chosen as

$$\begin{bmatrix} \ln \phi_{11} & & \\ & \ddots & \\ & & \ln \phi_{nn} \end{bmatrix}$$

Define an operator $P(t)$ by

$$\begin{aligned} P^{-1}(t) &= \Phi(t, 0)e^{-Rt} \\ \Phi(t, t_0) &= \Phi(t, 0)\Phi(0, t_0) \\ &= P^{-1}(t)e^{Rt}e^{-Rt_0}P(t_0) \\ &= P^{-1}(t)e^{R(t-t_0)}P(t_0) \end{aligned}$$

From (11.30),

$$\begin{aligned} x(t) &= \Phi(t, t_0)x(t_0) \\ &= P^{-1}(t)e^{R(t-t_0)}P(t_0)x(t_0) \end{aligned}$$

Put $z(t) = P(t)x(t)$, then

$$\begin{aligned} P^{-1}(t)z(t) &= P^{-1}(t)e^{R(t-t_0)}P(t_0)P^{-1}(t_0)z(t_0) \\ z(t) &= e^{R(t-t_0)}z(t_0) \end{aligned}$$

This equation has a time-invariant transition matrix and is the solution of the transformed version of (11.30),

$$z(t) = e^{R(t-t_0)}z(t_0) \tag{11.32}$$

This procedure is known as the *Floquet–Lyapunov transformation*.

11.7 Properties of the transition matrix

Identity property

$$\Phi(t_0, t_0)x(t_0) = x(t_0)$$

Inverse property If

$$\Phi(t, t_0)x(t_0) = x(t)$$

then

$$\Phi(t_0, t)x(t) = x(t_0)$$

or

$$\Phi(t_0, t)(\Phi(t, t_0)x(t_0)) = x(t_0)$$

$\Phi(t_0, t)$ is defined as the *inverse* of $\Phi(t, t_0)$, denoted $\Phi^{-1}(t, t_0)$. (For the time-invariant case, $\Phi^{-1}(t) = (e^{At})^{-1} = e^{-At} = \Phi(-t)$.)

Semi-group property

$$\begin{aligned}\Phi(t, t_1)\Phi(t_1, t_0)x(t_0) &= \Phi(t, t_1)(\Phi(t_1, t_0)x(t_0)) \\ &= (\Phi(t, t_1)\Phi(t_1, t_0))x(t_0) \\ &= x(t)\end{aligned}$$

Differentiation

$$\frac{d}{dt}(\Phi(t, t_0)) = A(t)\Phi(t, t_0)$$

11.8 Relation between the transfer-matrix description and the vector–matrix description

Assume that the process can be described by the equations

$$\left.\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}\right\} \tag{11.33}$$

and by the transfer matrix $G(s)$.

Assuming zero initial conditions in (8.33) and Laplace transforming, we obtain

$$y(s) = C(sI - A)^{-1}Bu(s) \tag{11.34}$$

from which, by uniqueness of solution, it follows that

$$G(s) = C(sI - A)^{-1}B \tag{11.35}$$

11.9 Equivalent systems

Two system representations

$$\begin{aligned} \dot{x} &= Ax + Bu \quad & \dot{z} &= Hz + Ju \\ y &= Cx & y &= Kz \end{aligned}$$

are said to be *equivalent* if

$$H = QAQ^{-1}, \quad J = QB, \quad K = CQ^{-1}$$

where Q is any $n \times n$ non-singular matrix with real elements. (We see at once that a diagonalised representation, as used, for instance, in Section 11.6.1, is equivalent to the original system with the modal matrix E playing the role of Q.)

The impulse responses of the systems are given by

$$\mathcal{L}^{-1}\{C(sI - A)^{-1}B\} \quad \text{and} \quad \mathcal{L}^{-1}\{K(sI - H)^{-1}J\}$$

or

$$C\ \mathrm{e}^{At}B \quad \text{and} \quad K\mathrm{e}^{Ht}J$$

However,

$$K\ \mathrm{e}^{Ht}J = CQ^{-1}Q\ \mathrm{e}^{At}Q^{-1}QB = C\ \mathrm{e}^{At}B \tag{11.36}$$

i.e. the impulse responses (and hence the transfer functions) of equivalent systems are identical. Thus, to one transfer function correspond many alternative state representations. A positive result is that $G(s)$ fixes the zero-state response but not the zero-input response.

The set $\{A, B, C\}$ is called a *realisation* of $G(s)$ if

$$C(sI - A)^{-1}B = G(s)$$

The realisation is *minimal* if there exists no other realisation $\{A', B', C'\}$ for which

$$\dim A' < \dim A$$

11.10 System realisation

Section 11.8 showed that, to a particular system realisation $\{A, B, C\}$, there corresponds a transfer matrix

$$G(s) = C(sI - A)^{-1}B$$

We now consider the converse problem. Given a particular $G(s)$, does there always exist a corresponding realisation $\{A, B, C\}$? If such a realisation exists, is it

unique? In general, how are realisations $\{A, B, C\}$ to be constructed from a knowledge of $G(s)$?

11.10.1 Existence

A realisation of $G(s)$ exists provided that $G(s)$ satisfies the following conditions:

(a) For each term of $G(s)$, the degree of the numerator does not exceed that of the denominator
(b) Each term of $G(s)$ is a rational function of s

11.10.2 Uniqueness

It is clear from the discussion on equivalent systems, Section 11.9, that the state space realisation corresponding to a particular $G(s)$ cannot be unique.

This leads us to ask the weaker question. Are all the realisations corresponding to a particular $G(s)$ equivalent to each other? More formally, does $G(s)$ define a unique equivalence class of realisations?

It is easy to show by counter-example that the answer to this last question is also no. However, it is true that $G(s)$ defines a unique equivalence class of minimal realisations.

11.11 Stability

A system is defined to be *asymptotically stable* if for any initial vector $x(t_0)$ and assuming zero input

$$||x(t)|| \to 0 \quad \text{as} \quad t \to \infty \tag{11.37}$$

where $||\ ||$ is a measure of the magnitude of the vector. One suitable measure is given by

$$||x|| = \left(\sum x_i^2\right)^{1/2} \tag{11.38}$$

A system is defined to be *stable* if given an initial vector $x(t_0)$ satisfying $||x(t_0)|| < \delta$, there exists an $\varepsilon > 0$ such that $||x(t_0)|| < \delta$ implies that

$$||x(t)|| < \varepsilon \quad \text{for all} \quad t \geq t_0 \tag{11.39}$$

11.11.1 Stability tests for continuous time systems

Let $\Phi(t)$ be the transition matrix of a system Σ whose state description is given by $\{A, B, C\}$. The system Σ is asymptotically stable if

$$||\Phi(t)|| \to 0 \quad \text{as} \quad t \to 0$$

The system Σ is stable if

$||\Phi(t)||$ remains bounded for all $t \geq 0$.

The system Σ is asymptotically stable if each eigenvalue λ_i of the matrix A satisfies

$$\Re(\lambda_i) < 0, \quad i = 1, \ldots, n$$

The system Σ is stable if each eigenvalue satisfies

$$\Re(\lambda_i) \leq 0, \quad i = 1, \ldots, n$$

11.12 Reachability, controllability, observability and reconstructibility for continuous time systems

Reachability A state x_1 is defined to be *reachable* if some control $u(t)$ applied over a finite interval (t_0, t_1), $t_1 > t_0$, transfers the system from an initial state $x(t_0) = 0$ to the state $x(t_1) = x_1$. A system is defined to be *reachable* if every state is reachable.

Controllability A state x_1 is defined to be *controllable* if some control $u(t)$ applied over a finite interval (t_0, t_1), $t_1 > t_0$, transfers the system from an initial state $x(t_0) = x_1$ to the state $x(t_1) = 0$. A system is defined to be *controllable* if every state is controllable.

Observability A state $x(t_0)$ of a system is *observable* if it can be determined from a knowledge of the values of the system outputs $y(t)$ over a finite interval (t_0, t_1), $t_1 > t_0$. A system is *observable* if every state is observable.

Reconstructibility A state $x(t_0)$ is *reconstructible* if it can be determined from a knowledge of the values of the system output $y(t)$ over a finite interval (t_1, t_0), $t_1 < t_0$. For continuous time systems, reachability is equivalent to controllability and observability is equivalent to reconstructibility. The definitions will be seen to differ when discrete time systems are under consideration.

11.12.1 Controllability and observability tests for continuous time, time-invariant systems by the matrices Q_c, Q_o

Define the matrices

$$Q_c = (B, AB, \ldots, A^{n-1}B)$$

$$Q_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Then the n-dimensional system with state space description $\{A, B, C\}$ is completely controllable (completely observable) if and only if rank Q_c (rank Q_o) $= n$.

The proof of this theorem is omitted. However, the analogous theorem for discrete time systems is proved.

11.12.2 Controllability and observability tests for continuous time, time invariant systems by the diagonalisation method

Given a system described by the equations

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{11.40}$$

Then provided that A has distinct eigenvalues, the system can be diagonalised by putting $x = Ez$ in (11.40), where E is the modal matrix (see Section 11.6.1), to yield

$$\left.\begin{aligned} \dot{z} &= E^{-1}Aez + E^{-1}Bu \\ y &= CEz \end{aligned}\right\} \tag{11.41}$$

Since the matrix $E^{-1}AE$ is diagonal, the controllability and observability of the transformed system (11.41) can be tested as follows (proof by inspection):

The system is uncontrollable if any row of the matrix $E^{-1}B$ consists solely of zero entries.

The system is unobservable if any column of the CE matrix consists solely of zero entries.

Controllability and observability are invariant properties under the transformation $x = Ez$. To see this, define matrices

$$\begin{aligned} Q_{c_1} &= (B, AB, \ldots, A^{n-1}B) \\ Q_{c_2} &= (E^{-1}B, (E^{-1}AE)E^{-1}B, \ldots, (E^{-1}AE)^{n-1}E^{-1}B) \end{aligned}$$

Since rank E^{-1} = rank $E = n$, then rank Q_{c_1} = rank Q_{c_2} and the invariance is proved.

Thus, the tests based on diagonalisation can be used to check whether the original system is uncontrollable or unobservable.

11.13 The unforced state equation in discrete time

The general (i.e. time-varying) unforced discrete time equation has the form

$$x(k+1) = A_k x(k), \quad x(0) \text{ given} \tag{11.42}$$

11.13.1 Existence and uniqueness of solution

The equation has one and only one solution for each integer k if and only if the matrices A_k are non-singular.

Proof

$$x(k) = A_{k-1}A_{k-2} \cdots A_0 x(0), \quad k \geq 0 \tag{11.43}$$

If any of the matrices A_i, $0 \leq i \leq k-1$ is singular, then by arguments based on the rank of mappings or otherwise, the solution is not unique—hence, it is necessary that all matrices A_i, $0 \leq i \leq k-1$ are non-singular.

It is also true that

$$x(k) = A_k^{-1}A_{k-1}^{-1} \cdots A_{-1}^{-1} x_0, \quad k \leq 0 \tag{11.44}$$

And from (11.43) and (11.44) it follows that non-singularity of the matrices A_k is sufficient to ensure that a unique solution exists.

11.13.2 The transition matrix for the discrete time equation

Define

$$\Phi(k, k_0) = A_{k-1}A_{k-2} \cdots A_{k_0}, \quad k > k_0 \tag{11.45}$$

and

$$\Phi(k_0, k_0) = I \tag{11.46}$$

The transition matrix can be used to generate solutions of (11.42) according to the relation

$$x(k) = \Phi(k, k_0)x(k_0), \quad k \geq k_0 \tag{11.47}$$

The transition matrix has the properties

$$\begin{aligned} \Phi(k_0, k_0) &= I \text{ (stated above and clearly necessary)} \\ \Phi(k_2, k_1)\Phi(k_1, k_0) &= \Phi(k_2, k_0), \qquad k_0 \leq k_1 \leq k_2 \end{aligned}$$

while if all the matrices A_k are non-singular for all k, then the expression holds for all k_0, k_1, k_2 regardless of their ordering.

For constant non-singular matrices A

$$\begin{aligned} \Phi(k - k_0) &= A^{k-k_0} \\ \Phi(k_1 + k_2) &= \Phi(k_1)\Phi(k_2), \quad \text{for all } k_1, k_2 \\ \Phi(k) &= \Phi^{-1}(-k) \end{aligned}$$

and in this case, $\Phi(k)$ can be calculated by inverse $\mathcal{Z}$ transformation as

$$\Phi(k) = \mathcal{Z}^{-1}\{zI - A^{-1}z\} \tag{11.48}$$

11.14 Solution of the discrete time state equation with forcing

Given the equation

$$x(k+1) = Ax(k) + Bu(k) \tag{11.49}$$

we can write successively

$$\begin{aligned} x(1) &= Ax(0) + Bu(0) \\ x(2) &= A(Ax(0) + Bu(0)) + Bu(1) \end{aligned}$$

until we can produce the desired result

$$x(k) = A^k x(0) + \sum_{j=0}^{k-1} A^j Bu(k-j-1) \tag{11.50}$$

In some ways this equation is rather easier than the equivalent continuous time equation. For instance, if we know A and B and are given a sequence $\{x(k)\}$, it is a simple matter to calculate a necessary control sequence $\{u(k)\}$ to steer the system through the given state sequence.

11.15 Obtaining the $\mathscr{Z}$ transform equivalent of the state equation

Given the single-input, single-output equations

$$\left.\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned}\right\} \tag{11.51}$$

$\mathscr{Z}$ transformation gives

$$\begin{aligned} zx(z) - zx(0) &= Ax(z) + Bu(z) \\ x(z) &= (zI - A)^{-1} zx(0) + (zI - A)^{-1} Bu(z) \end{aligned}$$

Compare with (11.50) and it can be seen that

$$\mathscr{Z}\{A^k\} = (zI - A)^{-1} z \tag{11.52}$$

But also from the definition of $\mathscr{Z}$ transformation,

$$\mathscr{Z}\{A^k\} = \sum_{k=0}^{\infty} A^k z^{-k} = I + Az^{-1} + A^2 z^{-2} + \cdots \tag{11.53}$$

From (11.51), we obtain

$$y(z) = Cx(z) \tag{11.54}$$

If we form the transfer function $y(z)/u(z)$, say $G(z)$, then in terms of the above equations, we have

$$G(z) = C(zI - A)^{-1}B \tag{11.55}$$

and if we use the series expansion of (11.55), then we obtain

$$G(z) = \sum_{k=1}^{\infty} Ca^{k-1}Bz^{-k} \tag{11.56}$$

11.16 Stability tests

Let a discrete time system Σ be described by the equation

$$x(k+1) = Ax(k) \tag{11.57}$$

Theorem The system Σ is asymptotically stable if each of the eigenvalues λ_i of A satisfies

$$|\lambda_i| < 1$$

Proof When A has distinct eigenvalues we can write (11.57) in terms of a new variable $q = E^{-1}x$, where E is the modal matrix, as

$$q(k) = (E^{-1}AE)^k q(0) = \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix} q(0) \tag{11.58}$$

and the statement is clearly true.
(The theorem applies also to the case where A has repeated eigenvalues. A modification is required to the proof, using the result that $||A^k|| \to 0$ as $k \to \infty$, if each eigenvalue of λ_i of A satisfies $|\lambda_i| < 1$.)

11.17 Reachability, controllability, observability and reconstructibility for discrete time systems

The properties of reachability, controllability, observability and reconstructibility are defined for discrete systems exactly as for continuous time systems (Section 11.12).

However, there are some important differences in the tests for these conditions between the discrete and continuous cases.

For instance, in a time-invariant continuous system, controllability implies reachability – a continuous controllable system can be transferred from any arbitrary state to any other arbitrary state in some interval of time. For a discrete time system to have the reachability property, the condition is also required that the system matrix A is non-singular.

If the discrete time equation has been obtained from the continuous time system equation, $\dot{x} = FX$, then since $A = e^{FT}$, A always satisfies $\det A \neq 0$. However, discrete time equations that represent inherently discrete phenomena do not necessarily satisfy this condition.

Consider the system

$$\left.\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned}\right\} \tag{11.59}$$

It has the solution

$$\begin{aligned} x(k) &= A^k x(0) + A^{k-1}Bu(0) + \cdots + ABu(k-2) + Bu(k-1) \\ &= A^k x(0) + (B, AB, \ldots, A^{k-1}B)(u(k-1), \ldots, u(0))^T \\ y(k) &= Cx(k) \end{aligned} \tag{11.60}$$

Test for reachability For reachability, set $x(0) = 0$, then any state in the n-dimensional state space can be reached in n steps provided that the vectors (B, $AB, \ldots, A^{n-1}B$) span the state space. This leads to the definition:

The system of equations is *reachable* if rank (B, $AB, \ldots, A^{n-1}B$) $= n$.
It also follows that the sequence

$$\begin{aligned} u(n-1) &= (B, AB, \ldots, A^{n-1}B)^{-1}(x(n) - A^n x(0)) \\ &\vdots \\ u(0) \end{aligned} \tag{11.61}$$

will transfer the single-input system from state $x(0)$ to state $x(n)$.

Test for controllability Referring to (11.60), we set $x(k) = 0$ and note that for controllability the equation that must be satisfied is

$$\begin{aligned} A^n x(0) &= -(B, \ldots, A^{n-1}B)(u(n-1), \ldots, u(0))^T \\ x(0) &= -(A^{-n}B, A^{-n+1}B, \ldots, A^{-1}B)(u(n-1), \ldots, u(0))^T \end{aligned} \tag{11.62}$$

The condition for controllability then follows as:

The system of equations is controllable if

$$\text{rank}(A^{-n}B, \ldots, A^{-1}B) = n \tag{11.63}$$

(*Note*: If the matrix A is non-singular, the conditions for controllability and reachability are identical.)

Tests for observability and reconstructibility The following conditions apply:

The system Σ is observable if the matrix

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has rank n.

The system is reconstructible if the matrix

$$\begin{bmatrix} CA^{-n} \\ CA^{-n+1} \\ \vdots \\ CA^{-1} \end{bmatrix}$$

has rank n.

Again, it can be seen that the two conditions are equivalent if the matrix A is non-singular.

11.18 Canonical state space representations

11.18.1 Introduction

Given a system Σ, we know that there are many possible state space representations $\{A, B, C\}$ corresponding with Σ. In a general sense, all state space representations can be thought of as representable by Figure 11.5.

Particular representations may arise in the first instance because of the physical background to the original problem. However, it is often advantageous to transform an arbitrary representation into one of the canonical representations that are to be described.

The advantages of the canonical representations are that they allow the system structure to be easily understood.

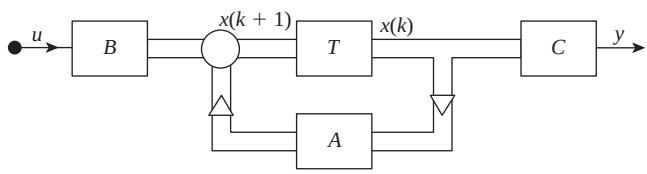


Figure 11.5 Discrete time state-variable representation

Let a single-input, single-output system Σ have some arbitrary but fixed state space representation that we designate by the suffix A.

$$\left.\begin{aligned} x_A(k+1) &= A_A x(k) + B_A u(k) \\ y(k) &= C_A x(k) \end{aligned}\right\} \tag{11.64}$$

Assume also that the system Σ can be represented by the difference equation

$$(z^n + a_{n-1}z^{n-1} + \cdots + a_0)y(z) = (b_{n-1}z^{n-1} + \cdots + b_0)u(z) \tag{11.65}$$

which can be written in the concise form

$$E(z, a)y(z) = F(z, b)u(z) \tag{11.66}$$

First we consider the simpler equation

$$E(z, a)q(z) = u(z) \tag{11.67}$$

E^{-1} can be realised in two alternative ways. They are illustrated in Figures 11.6 and 11.7 for a third-order version of (11.65). From Figure 11.7, it follows that

$$u(z) = (z^3 + a_2z^2 + a_1z + a_0)q(z)$$

Returning now to the general equation (11.66), it can be written either as

$$y(z) = F(z, b)(E(z, a)^{-1}u(z)) \tag{11.68}$$

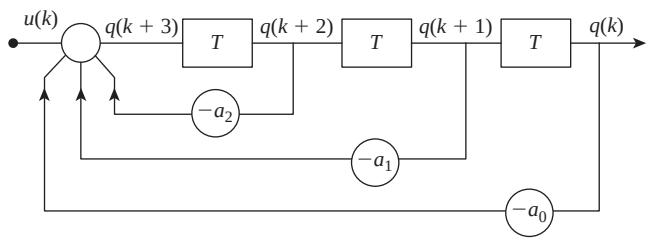


Figure 11.6 Realisation of (11.67) for a third-order system

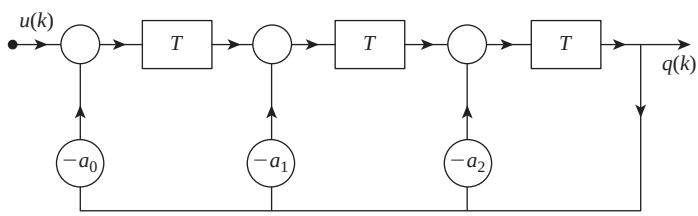


Figure 11.7 Alternative realisation of (11.67) for a third-order system

or as

$$y(z) = E(z,a)^{-1}(F(z,b)u(z)) \tag{11.69}$$

The alternative ways of realising E^{-1} in conjunction with (11.68) and (11.69) yield four different canonical realisations.

It should be understood that many alternative realisations are possible and that the names given here are by no means completely standard.

After the four canonical realisations have been given for single-input, single-output systems, the generalisation to multi-input, multi-output systems is considered. Finally, the Jordan canonical form is presented.

11.18.2 The reachability canonical form

From Figure 11.8 we can write

$$\begin{aligned} x_R(k+1) &= \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix} x_R(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(k) \\ &= A_R x_R(k) + B_R u(k) \end{aligned} \tag{11.70}$$

$$y(k) = C_R x_R(k) = (C_{R_1}, C_{R_2}, C_{R_3}) x_R(k) \tag{11.71}$$

To fix values for C_R we note from (11.65) that

$$y(z) = (b_2 z^2 + b_1 z + b_0) q(z)$$

Then by careful use of the figure it can be seen that

$$C_R = (b_2, b_1 - a_2 b_2, b_0 - a_1 b_2 + a_2^2 b_2 - a_2 b_1)$$

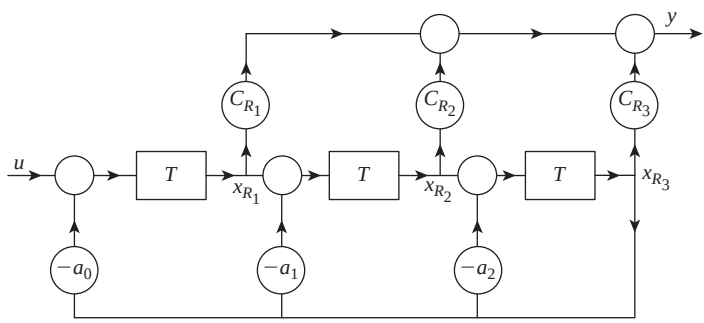


Figure 11.8 The reachability canonical form

The reachability matrix for an arbitrary system $\{A, B, C\}$ is given by

$$R = (B, AB, \ldots, A^{n-1}B)$$

and here by

$$R_R = (B_R, A_R B_R, \ldots, A_R^{n-1} B_R) = I \tag{11.72}$$

The substitution $x = Rx_R$ transforms the arbitrary system representation $\{A, B, C\}$ into the reachability form, i.e.

$$A_R = R^{-1}AR, \quad B_R = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad C_R = CR$$

The coefficients of the matrix A_R can be computed conveniently from the characteristic equation of the matrix A.

Example **Conversion of an arbitrary system to reachability form:** A system is described by the equations

$$x(k+1) = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -2 & -1 \\ 0 & 1 & -1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(k) = Ax(k) + Bu(k)$$

$$y(k) = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} x(k) = Cx(k)$$

To convert to the reachability form, A_R can be calculated from the relation, $A_R = R^{-1}AR$, where R is the reachability matrix $R = (B, AB, A^2B)$ of the given system. However, it is easier to calculate the characteristic equation and then to use (11.70):

$$(zI - A) = \begin{bmatrix} z & -1 & 0 \\ 1 & z+2 & 1 \\ 0 & -1 & z+1 \end{bmatrix}$$

leading to the characteristic equation

$$z^3 + 3z^2 + 4z + 1 = 0$$

Thus,

$$A_R = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -4 \\ 0 & 1 & -3 \end{bmatrix}, \quad B_R = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C_R = CR = (B, AB, A^2B)$$

$$= (\,1 \quad 0 \quad 1\,) \begin{bmatrix} 0 & 1 & -3 \\ 1 & -3 & 5 \\ 1 & 0 & -3 \end{bmatrix} = (\,1 \quad 1 \quad -6\,)$$

11.18.3 The controllability canonical form (phase-variable form) (Figure 11.9)

$$x_c(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} x_c(k) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k) \tag{11.73}$$

$$= A_c x_c(k) + B_c u(k)$$

$$y(k) = (b_0, b_1, b_2) x_c(k) \tag{11.74}$$

The controllability canonical form can be written down by inspection from a given difference equation.

We can transform an arbitrary representation $\{A, B\}$ into the controllability representation $\{A_c, B_c\}$ by inspection of the characteristic equation. From A, B, A_c, B_c we can then construct a transformation L between the two representations that

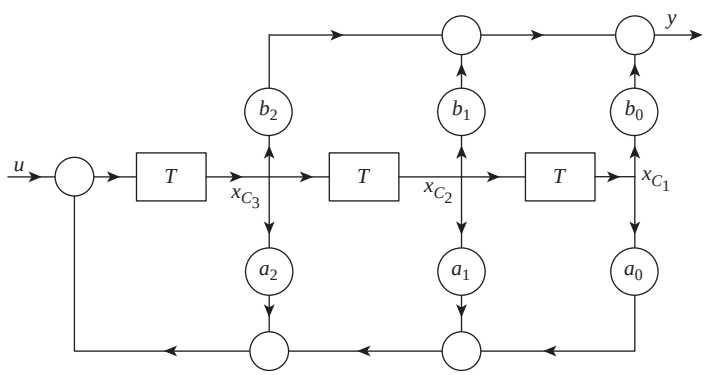


Figure 11.9 The controllability canonical form (phase-variable form)

we may use to obtain C_c, or to return a control design undertaken in terms of A_c, B_c to the real-world co-ordinates of the A, B system.

To proceed we form

$$Q_c = (B, AB, \ldots, A^{n-1}B)$$

$$Q_c' = (B_c, A_cB_c, \ldots, A_c^{n-1}B_c)$$

and seek a transformation L such that $x = Lx_c$. We note that

$$\begin{aligned} Ax(k) + Bu(k) &= x(k+1) = Lx_c(k+1) \\ &= LA_cx_c(k) + LB_cu(k) = ALx_c(k) + Bu(k) \end{aligned}$$

from which

$$A = LA_cL^{-1}, \quad B = LB_c$$

therefore,

$$Q_c = (LB_c, LA_cL^{-1}LB_c, \ldots) = LQ_c'$$

For a controllable single-input, single-output system, Q_c, Q'_c are square and invertible. Therefore, we can construct L or L^{-1} from the expressions

$$L = Q_c(Q_c')^{-1}, \quad L^{-1} = Q_c'Q_c^{-1}$$

Example Transform the system

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k) = Ax(k) + Bu(k)$$

to controllability canonical form and determine the transformation L, linking the original representation with the canonical form.

A_c can be written down from inspection of the characteristic equation

$$(z-1)(z+2) - 3 = z^2 + z - 5 = 0$$

We obtain immediately

$$x_c(k+1) = \begin{bmatrix} 0 & 1 \\ 5 & -1 \end{bmatrix} x_c(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$
$$= A_c x_c(k) + B_c u(k)$$

$$Q_c = (B, AB) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$Q_c' = (B_c, A_c B_c) = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$L = Q_c (Q_c')^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$L^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

We may check this by verifying that $A = LA_cL^{-1}$:

$$LA_cL^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} = A$$

Continuation of the example to show an approach to control system design Having reached this point, we cannot resist the temptation to continue to show how useful the controllability canonical form is in the design of feedback controllers, although we are being premature, since the topic is not treated until Section 11.20.

Suppose then that we have to use feedback by setting $u = Dx$ to ensure that the resulting closed loop system has poles at $z = 0$ and $z = 1$. We need the closed loop characteristic equation (recall that it is the same for all the different representations of the same system)

$$z(z-1) = 0$$

Let D_c be the necessary feedback matrix in the co-ordinates of the controllability form, then we can write down by inspection of the required closed loop equation

$$A_c + B_c D_c = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Since A_c, B_c are known, the elements d_1, d_2 of D_c must be chosen to satisfy

$$\begin{bmatrix} 0 & 1 \\ 5 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ d_1 & d_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

from which

$$D_c = (-5 \quad 2)$$

It now remains to determine D using the relation

$$D = D_c L^{-1} = (-5 \quad 2)\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = (-9 \quad 11)$$

Check

$$(A + BD) = \begin{bmatrix} -8 & 12 \\ -6 & 9 \end{bmatrix}$$

which has the characteristic equation

$$z^2 - z = 0$$

as required.

Notice carefully the line of reasoning that we have used:

(1) We are given a process description in the form of state space equations.
(2) We are required to devise a state-variable feedback strategy to place the closed loop poles at given locations.
(3) From (2) we can construct a characteristic equation having the required closed loop poles as its roots. (We are aware that the characteristic equation of a system is invariant under a similarity transformation.)
(4) We transform the process equations from (1) into a convenient canonical form.
(5) By inspection, we choose the coefficients of a feedback matrix D_c such that the required characteristic equation will be obtained for the closed loop system.
(6) D_c needs to be returned to the real-world original co-ordinates. We, therefore, determine a matrix L that can achieve the necessary transformation of D_c into the required feedback matrix D.
(7) We check that the matrix D, when used to provide feedback, does in fact achieve the desired effect.

11.18.4 The observability canonical form

See Figure 11.10 for a diagram depicting this form.

$$\begin{aligned} x_o(k+1) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} x_o(k) + \begin{bmatrix} b_{o1} \\ b_{o2} \\ b_{o3} \end{bmatrix} u(k) \\ &= A_o x_o(k) + B_o u(k) \end{aligned} \tag{11.75}$$

$$y(k) = C_o x_o(k) = (1, 0, 0) x_o(k) \tag{11.76}$$

The matrices A_o, B_o, C_o are related to the matrices $\{A_R, B_R, C_R\}$ in the reachability representation by

$$A_o = A_R^T, \quad B_o = C_R^T, \quad C_o = B_R^T$$

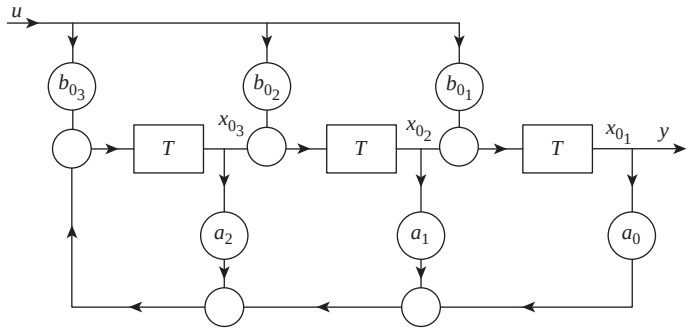


Figure 11.10 The observability canonical form

Define

$$D = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

then given an arbitrary representation $\{A, B, C\}$, the observability form can be obtained from the expressions

$$A_o = D^{-1}AD, \quad B_o = D^{-1}B$$

(C_o is given by (11.76)).

11.18.5 The reconstructibility canonical form

Refer to Figure 11.11, from which the representation below follows:

$$\begin{aligned} x_p(k+1) &= \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix} x_p(k) + \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} u(k) \\ &= A_p x_p(k) + B_p u(k) \end{aligned} \tag{11.77}$$

$$y(k) = (\,0 \quad 0 \quad 1)x_p(k) = C_p x_p(k) \tag{11.78}$$

The representation can be written down by inspection from a transfer function. Like the observability canonical form, it has the simplest possible output matrix. Other properties follow analogously to those of the three canonical forms described previously.

The four canonical forms described above have the considerable advantage that their patterns are simple and their coefficients always real.

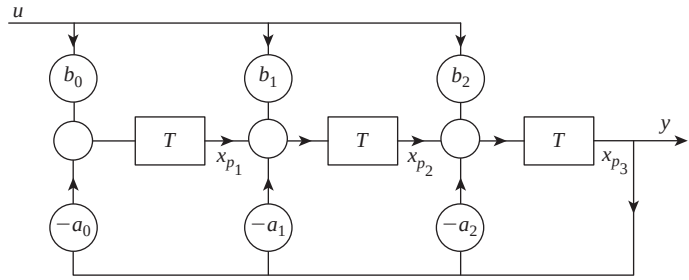


Figure 11.11 The reconstructibility canonical form

11.18.6 State equations for multi-input, multi-output processes

A process with n inputs and n outputs can be represented in discrete time by a set of n^2 difference equations linking the n input variables $u_1, \ldots, u_n$ with the n output variables $y_1, \ldots, y_n$. Define

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

then n^2 difference equations can be written with the aid of matrices $A_i, B_i\ (i = 1, \ldots, n)$ and the operator z in the form:

$$(Iz^n + A_{n-1}z^{n-1} + \cdots + A_0)y(z) = (B_{n-1}z^{n-1} + \cdots + B_0)u(z) \tag{11.79}$$

Analogous to the derivations for single-input, single-output systems, state equations can be written. For instance, analogous to the derivation in Section 11.18.3, we obtain

$$\begin{bmatrix} x_1(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -A_0 & -A_1 & \cdots & -A_{n-1} \end{bmatrix} \begin{bmatrix} x_1(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ I \end{bmatrix} \begin{bmatrix} u_1(k) \\ \vdots \\ u_n(k) \end{bmatrix} \tag{11.80}$$

$$y(k) = (B_0, \ldots, B_{n-1}) \begin{bmatrix} x_1(k) \\ \vdots \\ x_n(k) \end{bmatrix} \tag{11.81}$$

as a state representation in which (note) each of the x_i is itself an n vector.

Example The example will resolve any problems of interpreting the notation of this section. A process is described by the equations

$$\begin{aligned} y_1(k+2) + y_1(k) + y_2(k+1) &= u_1(k) + u_2(k+1) + 2u_2(k) \\ 2y_1(k) + y_2(k+2) &= u_1(k+2) + u_2(k) \end{aligned}$$

$\mathcal{Z}$ transforming produces the equations

$$\begin{aligned} z^2 y_1(z) + z y_2(z) + y_1(z) &= z u_2(z) + u_1(z) + 2u_2(z) \\ z^2 y_2(z) + 2y_1(z) &= z u_1(z) + u_2(z) \end{aligned}$$

In matrix form (as 11.80)

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z^2 + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \right) y(z) = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} z + \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right) u(z)$$

leading to the state space equations.

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{pmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ -\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{pmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

11.18.7 The Jordan canonical form

Suppose that a transfer function $G(z)$ can be decomposed into partial fractions to yield

$$G(z) = \alpha_0 + \sum_{i=1}^{n} \frac{\alpha_i}{z - z_i} \tag{11.82}$$

(α_0 is zero if the degree of the denominator exceeds that of the numerator).

Define a state vector x such that

$$y(z) = \alpha_0 u(z) + \sum_{i=1}^{n} \alpha_i x_i(z) \tag{11.83}$$

$$x_i(z) = \frac{1}{z - z_i} u(z) \tag{11.84}$$

which in the time domain becomes

$$x_i(k+1) = z_i x_i(k) + u(k), \quad i = 1, \ldots, n$$

or in vector–matrix form

$$x(k+1) = \begin{bmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & z_n \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(k) \tag{11.85}$$

$$y(k) = (\alpha_1, \alpha_2, \ldots, \alpha_n) x(k) + \alpha_0 u(k) \tag{11.86}$$

The z_i are the roots of the characteristic equation of $G(z)$. In case some of the roots are repeated, the form of the partial fractions is modified in a manner that is presumed well known to the reader.

For instance, if the real root z_1 has multiplicity p, then $G(z)$ has the expansion

$$G(z) = \alpha_0 + \frac{\alpha_1}{(z - z_1)^p} + \frac{\alpha_2}{(z - z_1)^{p-1}} + \cdots + \frac{\alpha_p}{(z - z_1)} + \frac{\alpha_{p+1}}{(z - z_2)} + \cdots + \frac{\alpha_n}{(z - z_m)} \tag{11.87}$$

Define

$$\begin{aligned} x_1(z) &= \frac{1}{z - z_1} x_2(z) + \frac{1}{(z - z_1)^p} u(z) \\ x_2(z) &= \frac{1}{z - z_1} x_3(z) + \frac{1}{(z - z_1)^{p-1}} u(z) \\ &\vdots \\ x_p(z) &= \frac{1}{z - z_1} u(z) \\ x_{p+1}(z) &= \frac{1}{z - z_2} u(z) \\ &\vdots \\ x_n(z) &= \frac{1}{z - z_m} u(z) \end{aligned}$$

In the time domain, the following vector–matrix equation corresponds to (11.87).

$$x(k+1) = \overbrace{\begin{pmatrix} z_1 & 1 & 0 & 0 & \cdots & 0 & & & \\ 0 & z_1 & 1 & 0 & \cdots & 0 & & & \\ 0 & 0 & z_1 & 1 & \cdots & 0 & & & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & & & \\ 0 & 0 & 0 & 0 & \cdots & z_1 & & & \\ & & & & & & z_2 & & \\ & & & & & & & \ddots & \\ & 0 & & & & & & & z_m \end{pmatrix}}^{\text{a } p \times p \text{ block}} x(k) + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \begin{matrix} p-1 \\ \text{zeros} \\ \\ u(k) \\ \\ \\ \end{matrix} \tag{11.88}$$

$$y(k) = (\alpha_1, \ldots, \alpha_n)x(k) + \alpha_0 u(k) \tag{11.89}$$

These diagonal or block-diagonal representations are called the *Jordan forms*. Jordan forms decouple the modes of a system and allow the dynamic behaviour of a system to be appreciated by inspection.

11.19 The state-variable approach to control system design

Suppose that a single-input, single-output process can be described by the discrete time state equations (referred to below as the model)

$$\left.\begin{aligned} x(k+1) &= Ax(k) + Bu(k)y(k) \\ &= Cx(k) \end{aligned}\right\} \tag{11.90}$$

The *control problem* can be defined in a very general way as follows: Determine a control sequence, $u(0), u(1), \ldots$ such that the output sequence $y(0), y(1), \ldots$ behaves in some desired manner. If y has to be kept constant, this defines a *regulator problem*. If y has to follow a preset trajectory, this defines a *tracking problem*. If the control sequence has to be chosen such that y behaves in some desired manner while, at the same time, some profit function $J(u, y)$ is maximised, this defines an *optimal control problem*.

All the above problems are *open loop control problems* in that the control sequence is determined from a prior knowledge of the model with no account taken of the actual system response.

In practice, no model is complete nor can its parameters be specified exactly. Realistic control can only be achieved by *feedback control* in which the control sequence $u(0), u(1), \ldots$ is calculated from a knowledge of the measured output sequence $y(0), y(1), \ldots$.

One group of powerful feedback control algorithms requires that the control sequence be calculated from a knowledge of the state vector $x(0), x(1), \ldots$. Such a

strategy is called a *state-variable feedback strategy*. It is central to state-variable control theory that if the model is linear and the cost function J is quadratic, then the optimal control sequence can be generated as a linear function of the state vector $x(k)$.

When the state vector x is not accessible or is corrupted by noise, but is required for feedback, then it can be estimated by a *state observer* (noise-free case) or a *state estimator* (noisy case). When both the model parameters and the state vector are to be estimated, a combined *state and parameter estimator* can be used.

11.20 Design to achieve a specified closed loop characteristic equation

Control techniques, some of which will be outlined below, allow, within the limits of physical realisability, any process whatsoever to be given any desired closed loop dynamic performance.

The techniques basically amount to pole-shifting by appropriately weighted state-variable feedback.

11.20.1 Control design based on state-variable feedback

Assume that a discretised single-input, single-output continuous process is described by the equations

$$\left.\begin{aligned} x(k+1) &= \Phi(T)x(k) + \Psi(T)u(k) \\ y(k) &= Cx(k) \end{aligned}\right\} \tag{11.91}$$

State-variable feedback consists in replacing all or part of the input vector u by a linear combination of states, Dx, where D is a diagonal matrix to be chosen. The equation becomes, assuming all of the vector u is replaced,

$$x(k+1) = \Phi(T)x(k) + \Psi(T)Dx(k) \tag{11.92}$$

The equation can be $\mathcal{Z}$ transformed (neglecting initial condition effects) to yield

$$(zI - \Phi - \Psi D)x(z) = 0 \tag{11.93}$$

Suppose now that the closed loop poles of the system are required to be at given locations $\alpha_1, \alpha_2, \ldots, \alpha_n$ in the z plane.

Control design then consists in choosing D so that

$$zI - \Phi - \Psi D = (z - \alpha_1)(z - \alpha_2)\ldots(z - \alpha_n) \tag{11.94}$$

We first note that, for an nth order process, a suitable matrix D always exists to allocate the poles to n arbitrary locations in the z plane, provided that the process is controllable.

The calculation is particularly easy in those cases where the process equations, (11.91), are in one of the canonical forms (Section 11.18), since then the characteristic equation can be written down by inspection. The following section makes this clear.

Let the system to be controlled be transformed into the controllability canonical form (Section 11.18.3):

$$x(k+1) = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & & -a_{n-1} \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(k) \tag{11.95}$$

State feedback is introduced by setting

$$u(k) = Dx(k) = (d_0, d_1, \ldots, d_{n-1})x(k) \tag{11.96}$$

The state equation then becomes

$$\begin{aligned} x(k+1) &= (A + BD)x(k) \\ &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & & & \cdots & 1 \\ (-a_0 + d_0) & & & \cdots & (-a_{n-1} + d_{n-1}) \end{bmatrix} x(k) \end{aligned} \tag{11.97}$$

The characteristic equation is

$$\begin{aligned} &\det\ (zI - (A + BD)) = 0, \\ &(a_0 - d_0) + \cdots + (a_{n-1} - d_{n-1})z^{n-1} + z^n = 0 \end{aligned} \tag{11.98}$$

The state feedback matrix D can be chosen to modify the characteristic equation as required. In particular, the closed loop poles can be located where required.

Example The process

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ -0.2 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

is to have a double pole at a $z = 0$ when under closed loop control. By comparing with (11.95), we find $a_0 = 0.2$, $a_1 = 1$. We require $z^2 = 0$ as the closed loop characteristic equation, i.e. we required $d_0 = 0.2$, $d_1 = -1$, to make $D = (0.2, -1)$. Check

$$A + BD = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and the object is accomplished. (See also the example in Section 11.18.3.)

Note that if the state feedback matrix D is chosen so that in (11.98)

$$a_i - d_i = 0, \quad i = 0, \ldots, n-1$$

then the characteristic equation of the closed loop system becomes

$$z^n = 0$$

and this would lead to so-called *finite settling time (dead-beat) control* where, in response to a step, the controlled process reaches a steady state in a finite number of sampling intervals and then stays exactly at that steady state at all later sampling times.

However, note carefully, that the process behaviour between sampling intervals is not modelled and it is easy to produce simplistic designs that exhibit perfect behaviour at sampling instants but that oscillate wildly between those instants.

11.20.2 Modal control by state feedback

Let the system to be controlled have distinct eigenvalues and be in the Jordan form (Section 11.18.7):

$$\begin{aligned} x(k+1) &= \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} x(k) + Bu(k) \\ &= \Lambda x(k) + Bu(k) \end{aligned} \tag{11.99}$$

Let state feedback be applied by setting

$$u(k) = Dx(k) \tag{11.100}$$

The state equation becomes

$$x(k+1) = (\Lambda + BD)x(k) \tag{11.101}$$

The closed loop eigenvalues can be moved by choice of the feedback matrix D. If the matrix BD is diagonal, then the closed loop eigenvalues or modes can be moved independently. This depends on the form of B. The technique loses much of its simplicity when generalised to multi-input, multi-output processes.

Notice finally that, in common with all design methods involving a transformation of form, the implementation must be considered with care. Either the control design must be transformed to real-world co-ordinates or measured state variables must first be transformed to Jordan form and the calculated control vector must be inverse transformed before application to the actual system.

Worked example 11.1

A system has the equation

$$\left.\begin{aligned} \ddot{y} + 8\dot{y} + 7y = 0, \quad & y(0) = 100 \\ & \dot{y}(0) = 0 \end{aligned}\right\} \tag{W1}$$

Setting $x_1 = y, x_2 = \dot{y}$, express the system in state variable form

$$\dot{x} = Ax \tag{W2}$$

Determine the eigenvalues of the system. Is the system stable? Give reasons. Is the system oscillatory? Give reasons.

Determine the transition matrix for the system.

With the help of the transition matrix or otherwise, determine the time T^* at which x_2 takes on its greatest absolute value.

Using a time step of 1 s and again using the transition matrix, express (W2) in the form

$$x(k) = Px(k-1) \tag{W3}$$

What is the relation between the eigenvalues of the matrix P and those of the matrix A? Check your assertion numerically.

Use a time step of T^* to recalculate P for (W3) and hence determine the maximum absolute velocity.

Return to the original (W1). Solve it analytically and use the result to sketch the transient behaviour of y and $\dot{y}$.

Worked solution 11.1

$x_1 = y$, therefore $\dot{x}_1 = \dot{y} = x_2$ and $\ddot{y} = \dot{x}_2$. Equation (W1) can be written as two equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -7x_1 - 8x_2 \end{aligned}$$

where

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -7 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$\dot{x} = Ax$$

This is the system in state-variable form.

To find the eigenvalues, we need to solve the characteristic equation

$$\det\ (\lambda I - A) = 0$$

Now

$$\begin{aligned} \lambda I - A &= \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -7 & -8 \end{bmatrix} \\ &= \begin{bmatrix} \lambda & -1 \\ 7 & \lambda + 8 \end{bmatrix} \end{aligned}$$

with determinant $\lambda^2 + 8\lambda + 7$ and the characteristic equation is

$$\lambda^2 + 8\lambda + 7 = 0$$

Hence, the eigenvalues are the solutions of this equation, i.e.

$$\lambda_1 = -1, \quad \lambda_2 = -7$$

Since both eigenvalues have negative real parts, the system is asymptotically stable.

Since no eigenvalue has an imaginary component, the system is not oscillatory. Each eigenvector e_i must satisfy $(\lambda_i I - A)e_i = 0$ by definition. So we must have putting $\lambda_i = \lambda_1 = -1$,

$$\begin{bmatrix} -1 & -1 \\ 7 & 7 \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{21} \end{bmatrix} = 0$$

or

$$e_{11} - e_{12} = 0$$

Put $e_{11} = 1$ then $e_{21} = -1$. Similarly, putting $\lambda_i = \lambda_2 = -7$, we obtain

$$\begin{bmatrix} -7 & -1 \\ 7 & 1 \end{bmatrix} \begin{bmatrix} e_{12} \\ e_{22} \end{bmatrix} = 0$$

or

$$-7e_{12} - e_{22} = 0$$

putting $e_{12} = 1$ yields $e_{22} = -7$.

The modal matrix

$$E = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -7 \end{bmatrix}$$

The inverse of E is found to be

$$E^{-1} = \frac{1}{6} \begin{bmatrix} 7 & 1 \\ -1 & -1 \end{bmatrix}$$

Now

$$\Phi(t) = E \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} E^{-1}$$

(see Section 11.6.1). Hence,

$$\Phi(t) = \begin{bmatrix} 1 & 1 \\ -1 & -7 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-7t} \end{bmatrix} \begin{bmatrix} 7 & 1 \\ -1 & -1 \end{bmatrix} \frac{1}{6}$$

is the required transition matrix.

Considering only x_2, we have

$$x_2(t) = \frac{1}{6}(\phi_{21}(t)x_1(0) + \phi_{22}x_2(0))$$

But $x_2(0) = \dot{y}(0)$ is zero in this question. Hence,

$$x_2(t) = \frac{1}{6}(-7e^{-t} + 7e^{-7t})x_1(0)$$

and we can find maxima or minima by differentiating, i.e. setting

$$\dot{x}_2(t) = \frac{100}{6}(7e^{-t} - 49e^{-7t})$$

to zero. This leads to the condition for maximum or minimum as

$$\begin{aligned} 7e^{-t} &= 49e^{-7t} \\ e^{-t} &= 7e^{-7t} \\ -t &= \ln 7 - 7t \\ 6t &= \ln 7 \\ t &= 0.3243 = T^* \end{aligned}$$

as required.

Putting $t = 1$ into the transition matrix yields

$$\Phi(1) = \begin{bmatrix} 0.42904 & 0.06116 \\ -0.428 & -0.06025 \end{bmatrix}$$

and this is the matrix P required by the question, since

$$x(k) = \Phi(1)x(k-1) = Px(k-1)$$

Suppose that A were put in diagonal form A', it would have the values

$$A' = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

and P would have the form

$$P' = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix}$$

since $t = 1$ in this particular case.

Now eigenvalues are invariant under diagonalisation; hence, we can make the statement that

$$\lambda_i(P) = e^{\lambda_i}(A') = e^{\lambda_i}(A)$$

in this case.

Check

The eigenvalues of P are 0.36788 and 9.1188×10^{-4} and these same values are found by calculating $e^{\lambda_1 t}$, $e^{\lambda_2 t}$ for $\lambda_1 = -1$, $\lambda_2 = -7$, $t = 1$.

To recalculate P for a time step T^*, we use the relation

$$P(T^*) = \Phi(T^*) = \frac{1}{6}\begin{bmatrix} 4.958 & 0.61973 \\ -4.338 & 7.96 \times 10^{-5} \end{bmatrix}$$
$$= \begin{bmatrix} 0.8263 & 0.1033 \\ -0.7230 & 1.327 \times 10^{-5} \end{bmatrix}$$

The absolute maximum velocity is then found as

$$x_2(T^*) = [-0.723 \quad 1.327 \times 10^{-5}]\begin{bmatrix} 100 \\ 0 \end{bmatrix}$$
$$= -72.3$$

The analytical solution of (W1) is found to be

$$y = -16.67e^{-7t} + 116.67e^{-t}$$
$$\dot{y} = (7 \times 16.67)e^{-7t} - 116.67e^{-t}$$

The transient behaviour is sketched below in Figure 11.12.

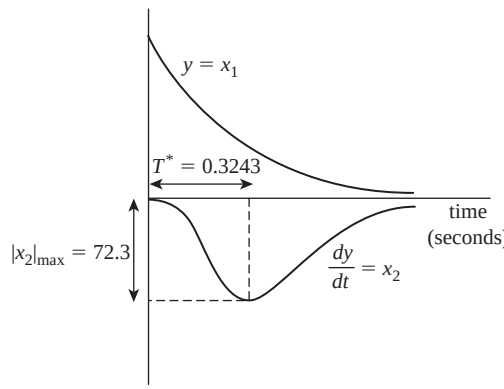


Figure 11.12 The initial response of the system described by equation W1

Worked example 11.2
A single-input, single-output process described by the equations

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

is subjected for a long time to the input signal shown in Figure 11.13, which switches between zero and unit magnitude.

Let y^*, y_* be the upper and lower limits, respectively, reached in the steady state by the response $y(t)$. Using the transition matrix, derive expressions for y^* and y_*.

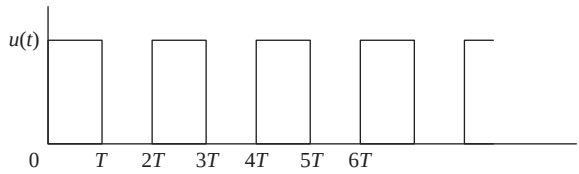


Figure 11.13 The input sequence

Calculate y^* and y_* for the process with

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [\,1 \quad 0\,], \quad T = 1$$

Worked solution 11.2
Since a long time has elapsed, it can be assumed that the upper and lower values are now constant and that the following relations hold

$$x_* = \Phi(T)x^*$$
$$x^* = \Phi(T)x_* + \Psi(T)u$$

Substituting and setting $T = 1$, $u = 1$

$$x_* = \Phi(1)[\Phi(1)x_* + \Psi(1)]$$
$$(I - \Phi(1)^2)x_* = \Phi(1)\Psi(1)$$
$$x_* = (I - \Phi(1)\Phi(1))^{-1}(\Phi(1)\Psi(1))$$
$$\Phi(1) = \exp(A) = \begin{pmatrix} 0.6004 & 0.2325 \\ -0.4651 & -0.0972 \end{pmatrix}$$
$$\Psi(1) = A^{-1}(\Phi(1) - I)B = \begin{pmatrix} 0.1998 \\ 0.2325 \end{pmatrix}$$
$$x_* = \begin{pmatrix} 0.2093 \\ -0.1497 \end{pmatrix}$$
$$x^* = \Phi(T)^{-1}x_* = \begin{pmatrix} -1.9525 & -4.6708 \\ 9.3415 & 12.0598 \end{pmatrix}x_* = \begin{pmatrix} 0.2907 \\ 0.1497 \end{pmatrix}$$

Since $y = Cx = [1\ 0]x$, then

$$y_* = 0.2093, \quad y^* = 0.2907$$

See Figure 11.14 where the steady state response is sketched.

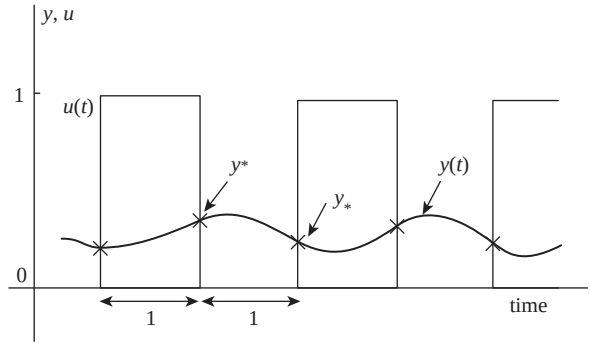


Figure 11.14 Solution to worked example 11.2

11A It seems remarkable that an oscillatory solution can be generated by repeated multiplication by a constant matrix (Figure 11.15)

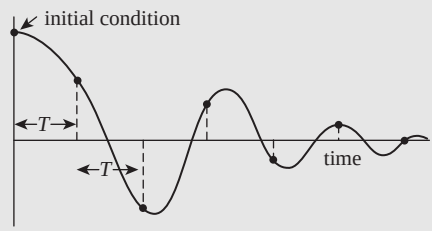


Figure 11.15 The points marked on the transient solution form a time series that can be generated by repeatedly multiplying the initial condition vector by a constant transition matrix

11B Conservation of dimension under linear transformations

Let L: $P \to Q$ be a linear transformation from

$P = R^n$ to $Q = \mathbb{R}^n$

Dom L is defined as the subspace of P on which the transformation operates.

Range L is defined as the subspace of Q satisfying

$\text{range } L = \{Lx \,|\, x \in \text{dom } L\}$

Ker L is defined as the subspace of P satisfying

$$\ker L = \{x \mid Lx = 0\}$$

Then the conservation of dimension insists that

$$\dim(\text{range } L) + \dim(\ker L) = \dim(\text{dom } L)$$

This means that the dimensionality of the range of the transformation L may be less than the dimensionality of the domain. Such a situation will occur whenever dim (ker L) > 0.

This 'loss of dimension into the kernel' is exactly the mechanism by which a system becomes uncontrollable. Tests for controllability amount to tests for ensuring that dim (ker L) $= 0$, where the transformation L is constructed so as to represent the operation of mapping $x(0)$ into $x(t)$. In this we have

$$L(x(0), u(\tau), l) : x(0) \rightarrow x(t)$$

i.e. the mapping depends on $x(0)$ and on the particular function u defined on the interval $I = [t_0, t]$. The actual tests for controllability have been derived from linear algebra. See Chen (1984) for detailed descriptions of the techniques.

In a system that is not controllable, there are some states that cannot be reached in finite time by any control strategy. In fact, some subsets of the state ((iii) and (iv) in Figure 11.16) cannot be influenced by the input.

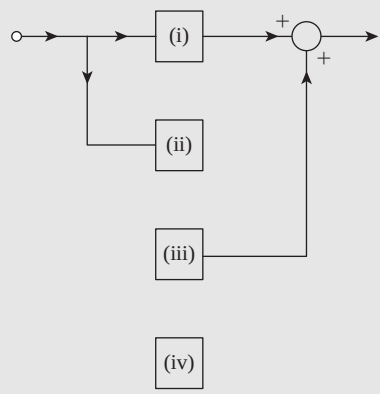


Figure 11.16 *Every linear system can be decomposed into four blocks: (i) controllable and observable; (ii) controllable but not observable; (iii) observable but not controllable; (iv) neither controllable not observable*

Observability is a dual of controllability. It is concerned with the question: Does measurement of the output y of a system allow complete knowledge of the state vector to be determined?

An interesting view, due to Kalman, sees every system as representable by four blocks. The idea is illustrated in Figure 11.16.

11C A deeper look at controllability and observability

A celebrated paper of R.E. Kalman (1962) demonstrates that the state variables of an linear time-invariant system can be separated into four disjoint categories, those states that are

(a) controllable and observable
(b) controllable but not observable
(c) observable but not controllable
(d) neither controllable nor observable

Here we will look only at the states in category (a) and ask the Orwellian question: Are all the states in category (a) equally controllable and observable, or are some states 'more controllable and observable' than others?

To investigate further, we recall that the state equation

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

may be transformed into a variety of different forms, by the substitution

$$z = Px$$

where P is any invertible matrix.

With this transformation, the state equation then becomes

$$\dot{z} = P^{-1}APz + PBu$$

$$y = CP^{-1}z$$

In earlier parts of Chapter 11, such transformations were used to produce different canonical forms but here we have a different goal; we seek a transformation that will allow the relative importance of the states in category (a), in terms of input/output energy or information transfer, to be displayed.

To proceed, we introduce two matrices: **the controllability Gramian** W_c defined by

$$Wc = \int_0^\infty \exp(At)BB^T \exp(A^T t)dt$$

and **the observability Gramian** W_o defined by

$$Wo = \int_0^\infty \exp(A^T t)C^T C \exp(At)dt$$

As an aside, let us note that these are well-established matrices in control theory and they have the following significance.

If the matrix W_c is non-singular, i.e. has full rank for every $t > 0$, then the pair (A, B) is controllable.

If the matrix W_o is non-singular, i.e. has full rank for every $t > 0$, then the pair (A, C) is observable.

The next step to our goal is to note that, remarkably, there exists an invertible matrix P such that

$$PWcP^T = P^{-T}WoP^{-1} = \sum = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \sigma_n \end{pmatrix}$$

The individual entries in the matrix $\sum$ are called Hankel singular values and their relative magnitudes represent the relative significance of each of the states in category (a), in an input/output sense. (Noting though that in a real application care would need to be taken that unbalanced scaling of individual variables did not invalidate the results).

See Glad (2000) Chapter 3 and Damm and Homeyer (2010) for proofs and further background.

Chapter 12

Links between state space and classical viewpoints

12.1 Introduction

The purpose of this chapter is to increase intuitive understanding of state space representations by linking to situations that are already familiar in a classical transfer function or frequency response setting.

(It has to be admitted that it is easier to see meaning in a curve showing frequency response than it is to understand several rectangular arrays of numbers or symbols.)

However, although frequency response methods are quickly understood, the foundations of those methods are quite involved and require a knowledge of the theory of functions of a complex variable and an understanding of Cauchy's integral theorem and Laurent's series.

The first link is an illustration of how the numerical entries in a system's A matrix can, intriguingly, be identified with the frequency-dependent rectangular co-ordinates of points on the corresponding (inverse) Nyquist diagram.

Two well-known textbook control systems; position control and cascade control, are analysed by state space methods. The ease of modelling and the power of the analysis in the state space domain are apparent and the linkage (state space to classical) quite obvious.

Modal analysis of state space systems is accomplished through eigenvalue/eigenvector analysis and can be nicely linked to the individual fundamental physical behavioural components of a time response as well as to the component trajectories in a phase space representation of system responses.

All of the various possible system representatives (time responses, frequency responses, pole–zero representations, transfer functions, diagrammatic representations, state equations, sets of linear differential equations) can be linked conceptually and computationally through a passive central core that lists all the qualitative and numerical data necessary to completely specify any finite-dimensional linear system that is describable by sets of ordinary different equations. Of course this is the basis for the operation of Matlab, Scilab, Octave etc. and is immensely useful for practical application and for learning the subject. We illustrate this aspect with a few examples.

This material, by linking ideas of single-input, single-output (SISO) systems with matrix representations and applications of eigenvalues/eigenvectors, is intended to help readers make the transition from classical frequency response ideas into the less intuitive but very powerful matrix-dominated world of multivariable systems.

12.2 A state space view of cascade control

Cascade control finds valuable application in certain situations where a relatively fast inner loop sits inside a slower outer loop (Figure 12.1). In classical terms, cascade control may allow the inclusion of an additional measurement or an additional actuator into what is seen as essentially a SISO system. Here we take a state variable view and illustrate the power and flexibility of that approach. For concreteness, we choose a jacketed reactor (Figure 12.2) as a specific example.

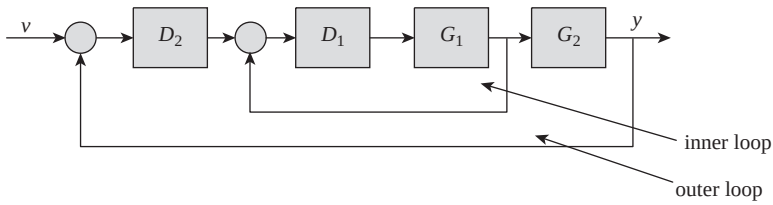


Figure 12.1 The usual classical control viewpoint of cascade control

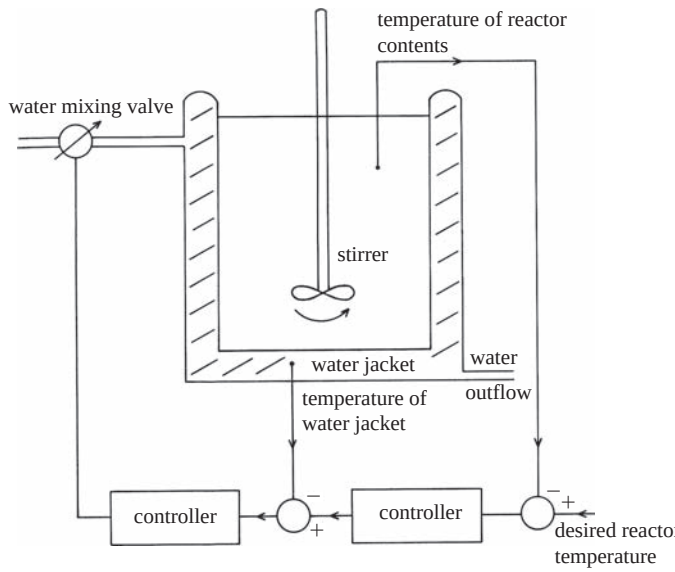


Figure 12.2 A jacketed reactor

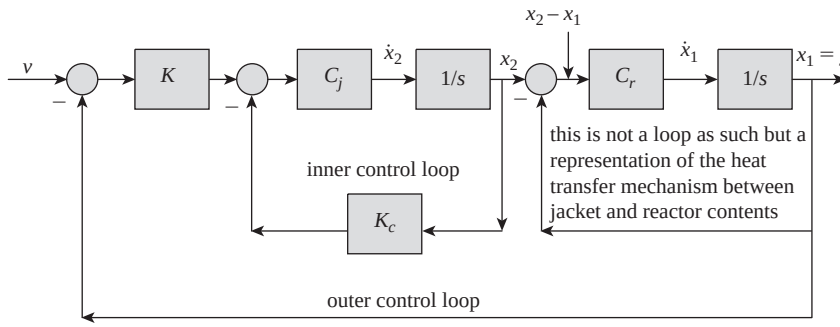


Figure 12.3 A simple model of a jacketed reactor

Here the water jacket temperature is the intermediate variable being controlled in the inner loop with the temperature of the reactor contents being controlled by the outer loop.

The right loop is virtual only and is a simple invention to ensure that heat flows in response to a temperature gradient, meaning that jacket and reactor contents tend towards the same equilibrium temperature. (The reason for placing the gain K_c in the feedback, rather than the forward path, is to make it easy to put the cascade loop out of action by simply setting $K_c = 0$ when required.)

12.2.1 Establishing the state space equations by inspection from the block diagram

Figure 12.3 shows the simple model of a jacketed reactor, with the water jacket J having a thermal resistance and heat capacity C_j and the reactor interior, with contents, having a thermal resistance and heat capacity C_r (we shall assume that $C_j = 5C_r$, i.e. the jacket can respond five times faster than the reactor contents). For this illustration, we assume that the two controllers consist only of scalar gains and initially, we shall assume that $K_c = 0$, that is we shall first see how the system performs without benefit of the inner loop. The only unusual feature of this model is the right-hand loop that ensures that heat transfer from jacket to reactor contents takes place at a rate proportional to their temperature difference, $x_2 - x_1$: this is not quite the normal transfer function for a cascaded reactor but the state space approach is very flexible indeed as the following analysis will show.

From the diagram,

$$\begin{aligned}\dot{x}_1 &= C_r(x_2 - x_1) = -C_r x_1 + C_r x_2\\ \dot{x}_2 &= C_j(Kx_3 - K_c x_2)\\ &= C_j(K[v - x_1] - K_c x_2)\\ &= -KC_j x_1 - K_c C_j x_2 + KC_j v\end{aligned} \tag{12.1}$$

leading to the state space matrices

$$\begin{aligned} A &= \begin{pmatrix} -C_r & C_r \\ -KC_j & -K_cC_j \end{pmatrix} \\ B &= \begin{pmatrix} 0 \\ KC_j \end{pmatrix} \end{aligned} \tag{12.2}$$

and since we shall want to monitor both state variables, we shall choose $C = (1\ 1)$, meaning we shall have $y_1 = x_1$, $y_2 = x_2$.

Inserting the numerical values $C_r = 1$, $C_j = 5$, $K_c = 0$ (for the first part of this demonstration there will be no inner loop feedback), we next seek a value for the main loop gain K, such that a reasonable damping factor, say, $\zeta = 1/\sqrt{2}$ results.

This requires a small side calculation.

From the two state equations that we derived,

$$\begin{aligned} \ddot{x}_1 &= -C_r\dot{x}_1 - C_r\dot{x}_2 \\ &= -C_r\dot{x}_1 - C_r\{-KC_jx_1 - K_cC_jx_2 + KC_jv\} \end{aligned} \tag{12.3}$$

Initially $K = 0$, so we can write

$$\ddot{x}_1 + C_r\dot{x}_1 + KC_jC_rx_1 = KC_jC_rv \tag{12.4}$$

and inserting the chosen values yields

$$\ddot{x}_1 + \dot{x}_1 + 5Kx_1 = 5Kv \tag{12.5}$$

Comparing this equation term by term with the canonical second-order equation

$$(s^2 + 2\zeta\omega_n s + \omega_n^2)y = \omega_n^2 v \tag{12.6}$$

and inserting the chosen ζ value yields

$$2\zeta\omega_n = 2\frac{1}{\sqrt{2}}\omega_n = \sqrt{2}\omega_n \tag{12.7}$$

which should equal 1 (or, equivalently, $2\omega_n^2$ should equal 1) and ω_n^2 should equal $5K$, so we must have $5K = 1/2$, $K = 0.1$. Using that value yields the A, B, C matrices

$$A = \begin{pmatrix} -1 & 1 \\ -0.5 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}, C = (\,1 \quad 1\,) \tag{12.8}$$

Now, setting the value $\zeta = \frac{1}{\sqrt{2}}$ in a second-order linear differential equation should result in the eigenvalues of the equivalent A matrix in the state space representation having complex eigenvalues whose real and imaginary parts are equal.

The eigenvalues of $A = \begin{pmatrix} -1 & 1 \\ -0.5 & 0 \end{pmatrix}$ are found to be

$\lambda_{1,2} = -0.5 \pm j0.5$, having equal real and imaginary parts as expected.

12.2.2 Bringing the inner loop into the model

Recall that we set the main loop gain K at the value $K = 0.1$, chosen specifically to yield a particular damping factor ($\zeta = 1/\sqrt{2}$). We now propose to fix a non-zero value for K_c to bring the inner loop into operation. It is reasonable to expect both from physical and mathematical viewpoints that the effect of a non-zero K_c will be a stabilising influence (mathematically, the new A matrix of the enhanced system will have a non-zero a_{22} element on its leading diagonal – see interlude 12A for further discussion). Therefore, we propose to multiply the main loop gain by a factor of 10, to $K = 1$, and attempt to fix the value of inner loop gain K_c to bring the overall damping factor to $\zeta = 1/\sqrt{2}$, as in the first case.

Repeating the calculation above with K now set to unity, we find that a value of $K_c = 3$ will achieve the required damping factor. With the new values $K = 1$, $K_c = 3$ and the C_j, C_r values unchanged we arrive at the new A_1, B_1, C_1 matrices (where the subscript 1 indicates the control system with inner loop operational):

$$A_1 = \begin{pmatrix} -1 & 1 \\ -5 & -3 \end{pmatrix}, B_1 = \begin{pmatrix} 0 \\ 5 \end{pmatrix}, C_1 = \begin{pmatrix} 1 & 1 \end{pmatrix} \tag{12.9}$$

As before, we check the eigenvalues of matrix A_1, they are

$$\lambda_{1,2} = -2 \pm j2 \tag{12.10}$$

They have equal real and imaginary parts as expected for that particular choice of damping factor.

12.2.3 Looking at the performance of the jacketed reactor under control, first with the single loop and then, for comparison purposes, with the inner loop operational

To make a comparison, we have several choices, but real-world jacketed reactors usually operate in batch mode, e.g. melting waxy substances that will later be made into a product. It is therefore logical to look at the step response and to look at the temperature of both the jacket and the reactor contents during that period.

Here I have solved the pair of simultaneous differential equations

$$\dot{x} = Ax + Bv \tag{12.11}$$

and

$$\dot{x} = A_1x + B_1v \tag{12.12}$$

directly using the *Scilab routine ODE* with the desired reactor contents temperature $\boldsymbol{v}$ fixed in each case at a constant value to yield a steady state temperature of 100.

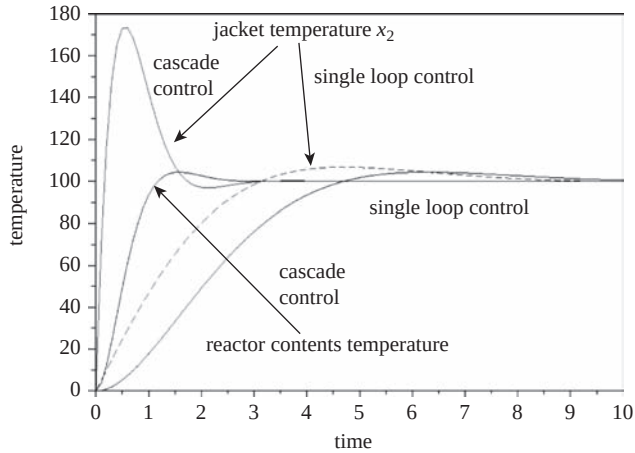


Figure 12.4 Transient behaviour of the jacketed reactor both without and with inner loop cascade control

The two graphs are plotted on the same axes and show the jacket temperature x_2 and the reactor contents temperature x_1 against time for single loop control and for control with the additional inner loop (so-called cascade control). Loop gains have been set as derived in the text above (Figure 12.4).

Comments on the above exercise: The model, although initially little more than a thought experiment, was quickly made into tractable physically related state equations that could generate potentially valuable time-domain results. A normal transfer function analysis of this SISO system might well have overlooked the extreme overshoot of the jacket temperature that is seen above.

(Naturally, if this was a real problem, the maximum temperature that was allowable and achievable in the reactor jacket would have to be carefully investigated.)

The naïve simplicity of the model hopefully makes it easy to appreciate the strengths of the approach.

12.3 An inverse Nyquist view of the entries in the *A* matrix of a system representation

Recall that the usual classical Nyquist approach to determining the expected closed loop behaviour of a dynamic system $G(s)$ is to make a polar plot of the *open loop response* of $G(j\omega)$ as a function of ω and then consider the shape of the plot, particularly in the region of the point $(-1, j0)$.

The inverse Nyquist approach is exactly similar with the difference that now the open loop response of the *inverse transfer function* $G(j\omega)^{-1}$ is plotted and again

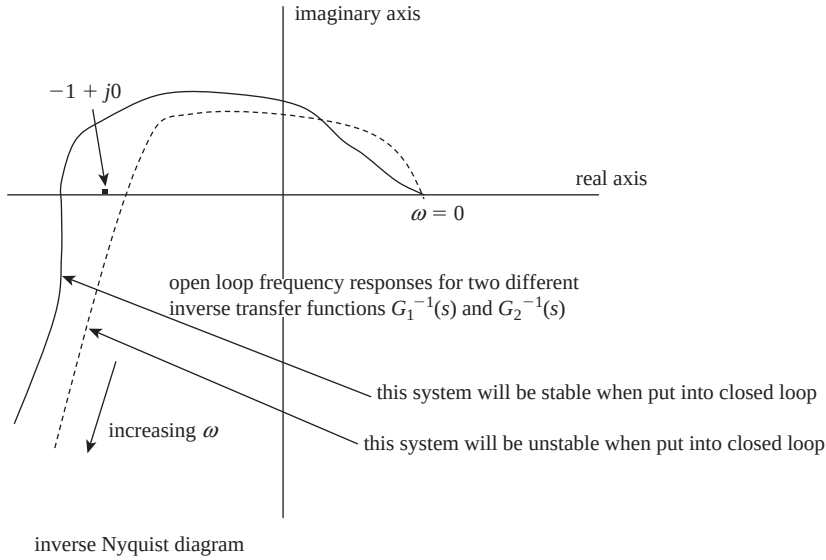


Figure 12.5 An inverse Nyquist diagram showing plots for two transfer functions $G_1^{-1}(s)$, $G_2^{-1}(s)$

is viewed with particular reference to the point $(-1, j0)$. Roughly speaking,* the system $G(s)$ will be stable provided that the inverse Nyquist plot does encircle the point $(-1, j0)$, otherwise it will be unstable (Figure 12.5).

Now consider the state space model for a SISO process

$$\dot{x} = Ax + Bu,\ y = Cx \tag{12.13}$$

$$\text{with } A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -l & -m & -p & -q \end{pmatrix}$$

(readers will realise that the structure of this A matrix is the same as the one used previously to model the jacketed reactor).

We now wish to show the frequency response significance of the four A matrix entries *l, m, p* and *q*.

First, we derive the equivalent transfer function

$$G(s) = \frac{1}{(s^4 + ls^3 + ms^2 + ps + q)} \tag{12.14}$$

* The full formal stability criterion involves encirclements and an appeal to the Cauchy integral theorem.

and its inverse

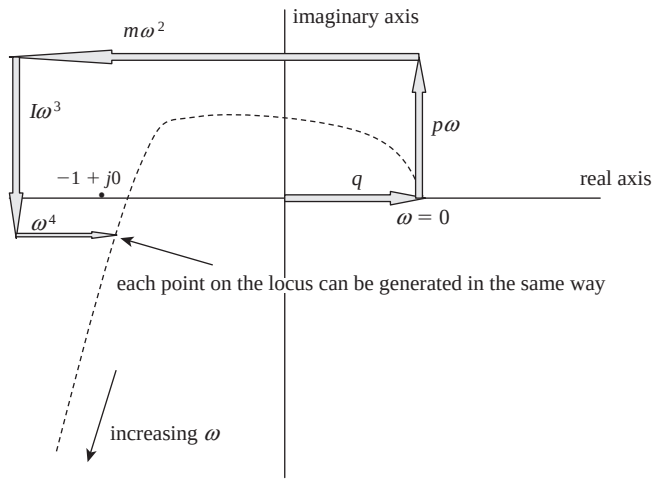
$$G(s)^{-1} = (s^4 + ls^3 + ms^2 + ps + q) \tag{12.15}$$

Next set $s = j\omega$ to obtain

$$\begin{aligned} &j^4\omega^4 + lj^3\omega^3 + mj^2\omega^2 + pj\omega + q \\ &= \omega^4 - jl\omega^3 - m\omega^2 + jp\omega + q \end{aligned} \tag{12.16}$$

and we see that the co-ordinates of every frequency-dependent point on the inverse Nyquist diagram can be directly identified with elements in the system's A matrix.

In the inverse Nyquist diagram that follows, I have attempted to make this interesting link clear (Figure 12.6).



this diagram shows how entries in a time-domain system matrix A can generate the frequency-dependent rectangular co-ordinates for every point on the same system's inverse Nyquist locus

Figure 12.6 Inverse Nyquist diagram for the system described above

12.4 Illustration of modes and modal analysis

Here we demonstrate how modal analysis can assist process understanding. Modal analysis makes use of the fact that all unforced transients in a linear system (i.e. arising from non-equilibrium initial conditions) must necessarily be a sum of individual modal responses.

It is well known to mechanical engineers that a large flexible structure may have many modes of vibration and that each of these possible modes can, in a possibly simplified analysis, be identified with a particular eigenvalue that corresponds with the frequency of oscillation of that mode of vibration.

However, modes and modal analysis do not need to be confined to mechanical systems and seeking out modes and identifying them with particular forms of dynamic behaviour can be very useful in understanding the transient behaviour of any linear system.

Figure 12.7 shows three connected water tanks with levels x_1, x_2, x_3 with the interaction ensuring that in the steady state and with no inflow or outflow, equilibrium will be reached with $x_1 = x_2 = x_3$.

Ignoring, for this particular illustration, all non-linearity and setting, for simplicity, all coefficients to unity we obtain:

Rate of increase of level x_1 = (difference in level between x_2 and x_1)

$$\dot{x}_1 = x_2 - x_1 \tag{12.17}$$

and similarly

$$\dot{x}_3 = x_2 - x_3 \tag{12.18}$$

with the central tank satisfying

$$\dot{x}_2 = (x_1 - x_2) + (x_3 - x_2) = x_1 - 2x_2 + x_3 \tag{12.19}$$

and as a vector matrix equation

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = Ax \tag{12.20}$$

We need to solve the above equation with the initial conditions (6, 14, −20) shown in the figure, i.e.

$$x(0) = \begin{pmatrix} 6 \\ 14 \\ -20 \end{pmatrix} \tag{12.21}$$

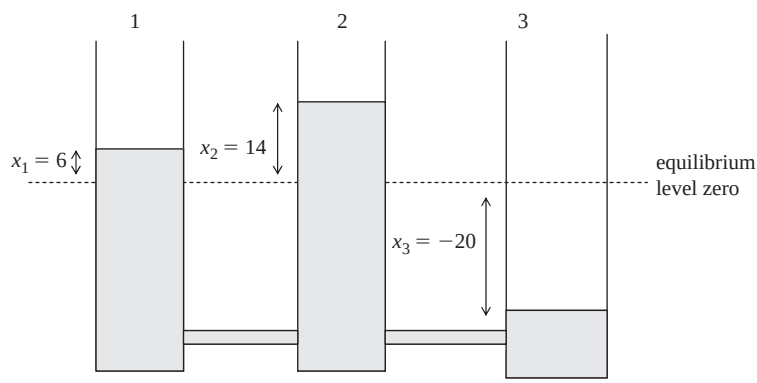


Figure 12.7 Three interconnected water tanks

We can solve the equation numerically using repeated multiplication by the transition matrix

$$x(n+1)T = \mathbf{\Phi}x(n)T,\ n = 0, 1, 2, \ldots \tag{12.22}$$

with a time step of (say) 0.1 s.

$\mathbf{\Phi}$ can be calculated using the Scilab/Matlab command {*expm(0.1*A)*} to yield

$$\mathbf{\Phi}(0.1) = \begin{pmatrix} 0.9092217 & 0.0863939 & 0.0043843 \\ 0.0863939 & 0.8272121 & 0.0863939 \\ 0.0043843 & 0.0863939 & 0.9092217 \end{pmatrix} \tag{12.23}$$

Note that an inspection of the terms in the transition matrix does not readily provide a feel for the dynamics of the process, although there is an unsurprising degree of *symmetry* and an obvious *diagonal dominance.*

The time behaviour is shown in Figure 12.8. Notice that the level in tank 1 initially rises before falling to the equilibrium level.

Modal analysis allowed us to break down the responses of a multivariable system into individual modes with physical meanings.

Here, the modal analysis of three interconnected water tanks revealed two dynamic modes:

(i) Mode number one involved both outer tank levels rising together with the central one falling twice as fast.
(ii) Mode number two involved the central level remaining stationary with the two outer levels moving in opposite directions.

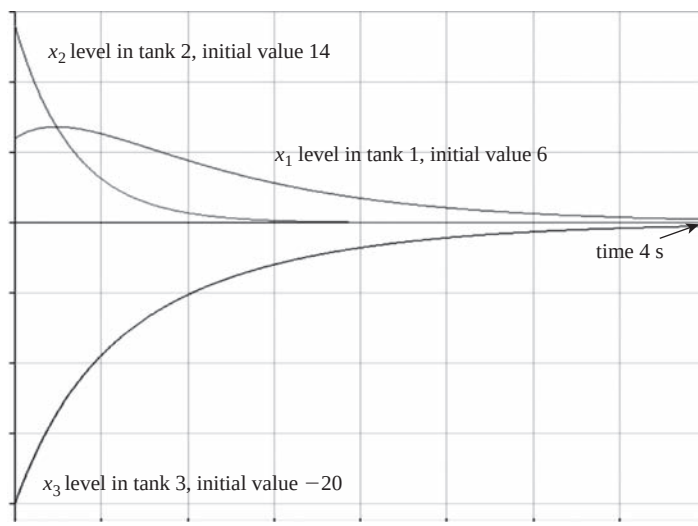


Figure 12.8 Transient response of the three-tank system

We require the eigenvalues and eigenvectors of the A matrix. Using Scilab, Matlab or similar software we obtain the eigenvalues as $\lambda_1 = -3$, $\lambda_2 = -1$, $\lambda_3 = 0$ with eigenvectors

$$e_1 = \begin{pmatrix} -0.408 \\ 0.816 \\ -0.408 \end{pmatrix}, e_2 = \begin{pmatrix} -0.707 \\ 0 \\ 0.707 \end{pmatrix}, e_3 = \begin{pmatrix} 0.577 \\ 0.577 \\ 0.577 \end{pmatrix} \tag{12.24}$$

Modal analysis involves representing the transient behaviour as a summation of these eigenvectors (which are by definition the modes of the system) varying in length with time as in the equation below, where the eigenvalues dictate the rate of decay or growth

$$x(t) = e^{\lambda_1 t} x(0)_{\text{mode 1}} + e^{\lambda_2 t} x(0)_{\text{mode 2}} + \ldots \tag{12.25}$$

The modal analysis viewpoint will now help us to completely understand the dynamic behaviour of the three-tank system.

To proceed, we set the initial conditions for modes 1 and 2 so that each one is a multiple of its related eigenvector (mode 3 will make no contribution since its eigenvalue is zero).

We must have $x(0)_{\text{mode 1}} + x(0)_{\text{mode 2}} = x(0) = \begin{pmatrix} 6 \\ 14 \\ -20 \end{pmatrix}$

and by inspection

$$x(0)_{\text{mode 1}} = \begin{pmatrix} -7 \\ 14 \\ -7 \end{pmatrix}, x(0)_{\text{mode 2}} = \begin{pmatrix} 13 \\ 0 \\ -13 \end{pmatrix} \tag{12.26}$$

and now the transient response of the system can be calculated as

$$x(t) = e^{\lambda_1 t} x(0)_{\text{mode 1}} + e^{\lambda_2 t} x(0)_{\text{mode 2}} \tag{12.27}$$

as plotted in Figure 12.9.

The above graph relates directly to the physics of our simple process – we can go further in that direction by realising that the two modes we have identified have a satisfying physical interpretation. Eigenvectors have arbitrary scaling (those shown earlier were normalised to have unit length), so for ease of understanding we rescale our eigenvectors to obtain

$$e_1 = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, e_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, e_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{12.28}$$

We now see (brilliant simplicity) how the system works. All possible transient responses have to be made up of the following:

(i) A fast mode, time constant 1/3 s, in which levels (recall – measured from equilibrium level) 1 and 2 are always equal to each other while level 2 is always twice as great and of opposite sign (Figure 12.10).

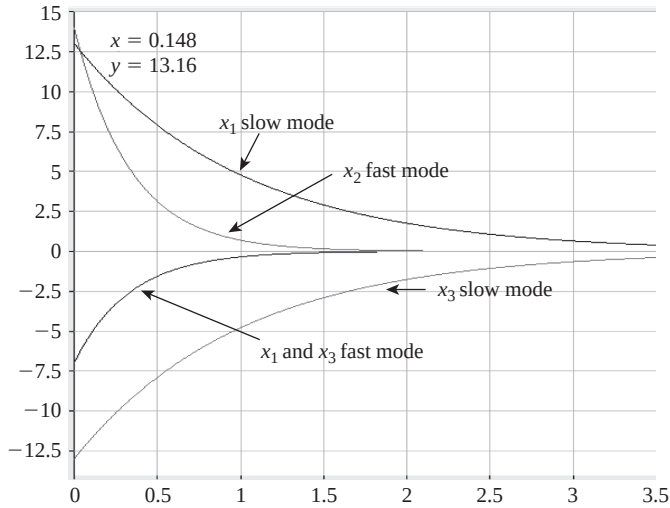


Figure 12.9 The dynamic response of the three-tank system to an initial disturbance in terms of the separate modes

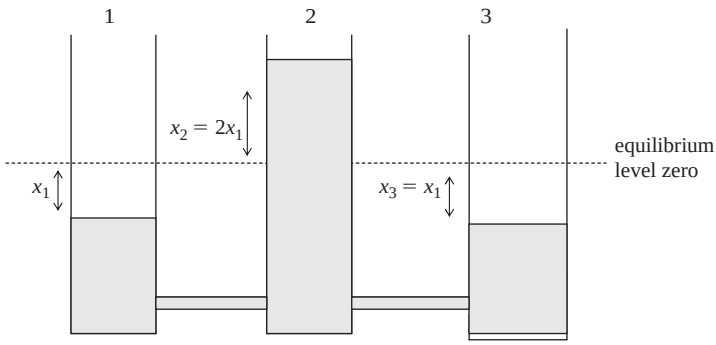


Figure 12.10 Mode one (the fast mode) for the three-tank system

(ii) A slow mode, time constant 1 s, in which levels 1 and 2 are always equal and opposite whereas level 2 remains at zero at all times (Figure 12.11).

It can be seen that the modal analysis of the three-tank system, as just performed, gives an excellent understanding of the possible forms of dynamic behaviour that the three-tank system can perform.

Finally, for completeness, displayed below is a plot of the transient response of the system and plot of the separate modal responses (Figure 12.12).

You may like to consider the following:

What, in general, does it mean to have a zero eigenvalue, a complex pair of eigenvalues and repeated eigenvalues?

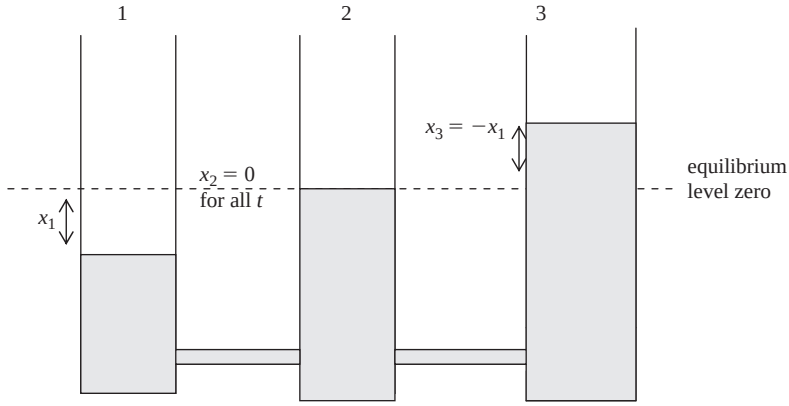


Figure 12.11 Mode two (the slow mode) for the three-tank problem

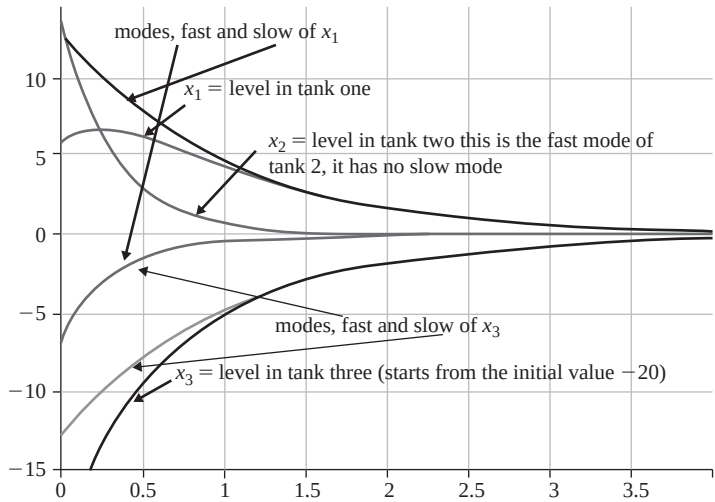


Figure 12.12 Complete display of the transient behaviour and of the separate modes

12A An interesting approach to modal analysis for a third order system with two complex eigenvalues

This Interlude demonstrates how to split the three dimensional state space of a third order system with complex eigenvalues to reveal the modal behaviour. Our starting point is example 34 from chapter 5 in (the still very worthwhile reference) Zadeh and Desoer (1963) and we use the book's rather nice

derivation, (noting though that the complex eigenvectors quoted in the book are unfortunately incorrect). However, following the Zadeh derivation but with corrected eigenvectors produced an interesting result with oscillatory second order behaviour in the u'/u'' plane.

The Zadeh and Desoer example is derived from an RLC circuit that has the following A matrix

$$A = \begin{pmatrix} -2 & 0 & -2 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix} \tag{12.29}$$

We calculate the eigenvectors E to produce the modal matrix (note – those vectors quoted here are correct):

$$E = \begin{bmatrix} 1.00000 + 0.00000i & -0.63246 + 0.00000i & -0.63246 - 0.00000i \\ 0.00000 + 0.00000i & 0.47434 - 0.27386i & 0.47434 + 0.27386i \\ 0.00000 + 0.00000i & 0.47434 + 0.27386i & 0.47434 - 0.27386i \end{bmatrix} \tag{12.30}$$

and the eigenvalues L are

$$\begin{bmatrix} -2.00000 + 0.00000i & 0.00000 + 0.00000i & 0.00000 + 0.00000i \\ 0.00000 + 0.00000i & -0.50000 + 0.86603i & 0.00000 + 0.00000i \\ 0.00000 + 0.00000i & 0.00000 + 0.00000i & -0.50000 - 0.86603i \end{bmatrix} \tag{12.31}$$

The transition matrix

$$\Phi = \begin{pmatrix} 0.13534 & 0.36177 & -0.70525 \\ 0.00000 & 0.12619 & -0.53351 \\ 0.0000 & -0.53351 & 0.65970 \end{pmatrix} \tag{12.32}$$

Now taking as initial condition a vector x_0, being a multiple of the first (real) eigenvector

$$x_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{12.33}$$

we find that the solution in three-space, a representative point of which must be

$$x(t) = \Phi x_0 = \begin{pmatrix} 0.13534 \\ 0.00000 \\ 0.00000 \end{pmatrix} \tag{12.34}$$

is a mode that is necessarily confined to a line (in this case the x_1 axis) along which it decays monotonically. This is a real mode of the system defined by the real eigenvector e_1.

We carry on to find the plane defined by the complex eigenvectors.

The first of the two complex eigenvectors has the form

$$\begin{pmatrix} -0.63246 + 0.00000i \\ 0.47434 - 0.27386i \\ 0.47434 + 0.27386i \end{pmatrix} \tag{12.35}$$

So using the approach advocated in Zadeh and Desoer whose justification (not reproduced here) requires using the reciprocal basis we define a plane in three-space by the two (real) basis vectors

$\boldsymbol{u}'$	$\boldsymbol{u}''$
[−0.63246]	[0.00000]
[0.47434]	[−0.27386]
[0.47434]	[0.27386]

Now, if we choose any initial condition vector lying in that u', u'' plane, then we shall find that, although this is a third-order system, only second-order behaviour is excited and that in the physical circuit, responding to the same specially chosen initial condition, all currents and voltages will have the same angular frequency, 0.866 rad/s, and the same exponential decrement, –0.5. This response is shown as curve (b) in Figure 12.13.

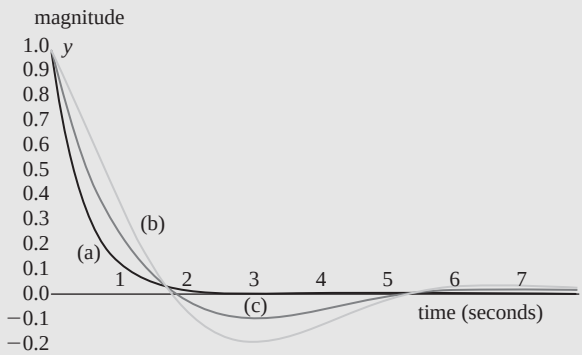


Figure 12.13 Three time responses of the system whose A *matrix is*

$$\begin{pmatrix} 2 & 0 & -2 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

: (a) the response of the real mode $y = e^{-2t}$ *to an initial condition on the* x_1 *axis; (b) the response of the oscillatory mode* $y = e^{-0.5t}(\cos 0.866t)$ *to an initial condition vector in the* u', u'' *plane, defined in the text above; (c) the*

response of the system to an arbitrary unity initial condition in three-dimensional state space. It has the form $y = \alpha e^{-0.5t}(\cos 0.866t) + \beta e^{-2t}$ α *and* β *being dependent on the direction of the chosen initial condition vector*

12.5 Moving between different system representations: the relationship between state space and transfer function representations

Although a transfer function contains the minimal parameterisation that can characterise the dynamic behaviour of a system, a state space representation of the same system may contain a great deal of additional structural information that does not necessarily have any effect on the input–output behaviour.

To see this, assume a particular system is described by the matrices $\{A, B, C, D\}$, where A is an $m \times m$ matrix.

That means that there are potentially m^2 entries $\{a_{ij}\}$ in the A matrix that can influence the input–output behaviour of the system. However, if the system is decomposed into four disparate parts (controllable and observable/controllable but not observable/observable or controllable/neither controllable nor observable), then it can be seen that only an $n \times n$ subset of the A matrix, representing the controllable and observable part, can affect the input–output behaviour, so only n^2 possible coefficients, $n \leq m$, need to be considered.

(This is not to suggest that response to initial conditions can always safely be ignored – consider, for instance, the dynamics of the planetary system of which the Earth forms a part.)

So continuing on, take the $n \times n$ subset of the A matrix, defined as above, and assume that it has n distinct eigenvalues λ_i, $i = 1, \ldots, n$, then it can be diagonalised by the manipulation

$$E^{-1}AE = \Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \tag{12.36}$$

where E is a square matrix whose columns are the n eigenvectors of A. Note that many authors, including this one, call E the *modal matrix* $E = (e_1|e_2|\ldots|e_n)$, so it is clear that Λ is in a sense a minimalised A matrix having n (diagonal) entries.

An important conclusion from the discussion above is that, to every transfer function $G(s)$, there correspond many state variable representations $\{A, B, C, D\}$ with the same input–output behaviour, and conversely, many state variable representations will have the same transfer function.

We can summarise this fact by thinking of a transfer function as containing sufficient information to fix the dynamic input–output behaviour (and

equivalently to fix the input–output response of a whole range of representations $\{A, B, C, D\}$).

The state representation $\{A, B, C, D\}$ on the other hand contains structural information, over and above that necessary to define the input–output response.

12.5.1 Poles and zeros of state space systems

To look further into the relation between transfer functions and state space representations, it is instructive to look at the definitions of poles and zeros for each. For a transfer function $G(s)$, poles are those values of s for which the magnitude of G approaches infinity whereas the zeros are those values for which the magnitude is zero.

Now considering the state space representation, we write the system equations in terms of the Rosenbrock system matrix

$$\begin{pmatrix} sI - A & B \\ -C & D \end{pmatrix} \begin{pmatrix} -x(s) \\ u(s) \end{pmatrix} = \begin{pmatrix} 0 \\ y(s) \end{pmatrix} \tag{12.37}$$

from which

$$(sI - A)x(s) = Bu(s) \tag{12.38}$$

$$Cx(s) + Du(s) = y(s)$$

$$x(s) = (sI - A)^{-1}Bu(s)$$

$$y(s) = [C(sI - A)^{-1}B + D]u(s)$$

so $G(s) = C(sI - A)^{-1}B + D$

Now recall that in one method of inverting the matrix $(sI - \boldsymbol{A})^{-1}$ by hand!, an intermediate stage is reached where we have

$$(sI - A)^{-1} = \frac{adj(sI - A)}{|sI - A|} \tag{12.39}$$

where *adj* indicates the adjoint of a matrix. Therefore, we can write

$$G(s) = C\frac{adj(sI - A)B}{|sI - A|} + D$$

$$G(s) = C\frac{adj(sI - A)B + |sI - A|D}{|sI - A|} \tag{12.40}$$

Examining the last equation, it is immediately obvious that only the matrix A can influence the poles of the whole system $\{A, B, C, D\}$, whereas all four of those matrices are potentially involved in determining the zeros.

The above equation hopefully explains why there are several different categories of zeros of state space systems, and thankfully, in practice, they do not need to be explicitly determined very frequently.

State space system zeros could be further investigated by considering the rank of the Rosenbrock system matrix, but that matrix is in general not square and its determinant is therefore, in general, not defined.

12.6 The relative gain array (RGA) and singular value decomposition: two useful tools to help in the task of controlling a multivariable process using a set of individual SISO control loops

Despite the proliferation of available design techniques for control of multivariable processes, it remains the case that industry often has a preference for using multiple single loop controllers, each one with a recognisable understandable task, rather than one monolithic multivariable controller.

Here we explain briefly two different approaches. First, the semi-empirical relative gain approach (RGA).

12.6.1 The RGA: in its simplest form, a forecaster of steady state interaction

The RGA, in its simplest form, is a forecaster of steady state interaction between possible loops in proposed designs to control an $m \times m$ process using m SISO loops.

This pragmatic method quantifies interaction and recommends which input u_i might best control output y_j for all possible i, j.

For a multivariable interacting process, a useful tool in the early stages of investigation is the concept of *relative gain*, introduced by Bristol (1966). Let a process in the steady state be represented by the equations

$$x_i = f_i(x_1, \ldots x_n, u_1, \ldots u_n), \quad i = 1, \ldots, n \tag{12.41}$$

The open loop, steady state sensitivity of state variable x_j to control input u_k is defined to be $[\frac{\partial x_j}{\partial u_k}]_O$.

The closed-loop steady-state sensitivity of state variable x_i to control input u_k is found by assuming that all other values of x_i, $i = 1, \ldots, n$, $i \neq j$ are held constant by closed loop operations. Substituting to take this into account yields the equation of closed loop operation. Differentiating the closed loop equation yields the closed-loop steady-state sensitivity of state variable x_j, to control input u_k, which is defined to be $[\frac{\partial x_j}{\partial u_k}]_C$.

The relative gain for the relation between u_k and x_j is then defined to be

$$\alpha_{jk} = \left[\frac{\partial x_j}{\partial u_k}\right]_O \Big/ \left[\frac{\partial x_j}{\partial u_k}\right]_C \tag{12.42}$$

The system described above has n^2 relative gains that can be arranged in a square matrix having the property that each column and each row sum to unity.

inspection of the matrix yields valuable information on the nature of the control problem and on the degree of interaction between variables. When $\alpha = 1$, there is no interaction between this particular control mechanism and any other. If all the values are positive, then clearly they all satisfy the relation $0 < \alpha < 1$, and under this condition, the system can be expected to be stable. Control loops will normally be chosen to operate between those variables linked by the largest values of α.

Negative values of α in general indicate a more difficult control problem with possibilities of instability for at least some of the possible control configurations.

12.6.2 Illustrative example

A process with state variables x_1 and x_2 has input controls u_1 and u_2 and is described in the steady state by the equations

$$\begin{aligned} x_1 &= u_1 + 0.5u_2 \\ x_2 &= -0.075u_1 + 0.1u_2 \end{aligned} \tag{12.43}$$

The closed loop equations are, for x_1, assuming that x_2 is held constant,

$$x_1 = u_1 + \frac{10x_2 + 0.75u_1}{2} = \frac{4}{3}[u_2 + 10x_2] + \frac{u_2}{2} \tag{12.44}$$

and for x_2, assuming that x_1 is held constant,

$$x_2 = -0.075\left[x_1 - \frac{u_2}{2}\right] + 0.1u_2 \tag{12.45}$$

It then follows that

$$\left[\frac{\partial x_1}{\partial u_1}\right]_O = 1, \quad \left[\frac{\partial x_1}{\partial u_1}\right]_C = 1.375, \quad \alpha_{11} = 0.727 \tag{12.46}$$

and the complete relative gain matrix may be determined from the property that rows and columns sum to unity.

The matrix of relative gains is therefore found to be

$$\begin{pmatrix} 0.727 & 0.272 \\ 0.272 & 0.727 \end{pmatrix} \tag{12.47}$$

In this example, control u_1 can control variable x_1 and control u_2 can control variable x_2 without stability problems being expected to occur. The interaction can be seen to degrade the control performance. Notice that dynamic effects were not considered at all.

Consider a different set of process equations in steady state

$$\begin{aligned} x_1 &= u_1 + 2u_2 \\ x_2 &= u_1 + u_2 \end{aligned} \tag{12.48}$$

This system yields the following matrix of relative gains.

$$\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \tag{12.49}$$

The minus signs indicate a change of polarity in the appropriate sensitivity coefficient between the open and closed loop conditions. More care is needed with this type of control problem than in the case where all the α coefficients are positive.

The decision may be made to design a decoupling controller to obtain a more favourable control situation. However, analysis shows that processes having relative gains outside the [0, 1] range and under decoupled control are excessively sensitive to small changes in parameters and may, in a practical situation, easily be brought into an unstable region by such small parameter changes (see Shinskey (1988)).

Notice that the RGA approach (restrictively) assumes that the process has the same number of actuators as sensors (a so-called square process).

Notice also that in an application of control to a process with strong but not destabilising interactions, the **pairing** of actuators with sensors will often be further influenced by speeds of response, with the fastest acting actuators being allocated to controlling the most critical process variables.

Matlab commands for RGA determination

The Matlab command

$$R = rga(G) \tag{12.50}$$

returns the RGA

while the command

$$R = rga(R) \tag{12.51}$$

applied repeatedly, eventually returns a matrix consisting only of zeros and ones with exactly one '1' in each row and in each column, this is called a 'selection matrix', S, by Matlab (since it implies that SISO control of a given multiple-input, multiple-output (MIMO) process can best be carried out by linking the inputs and outputs where indicated by the unity entries).

Both matrices R and S can be obtained by the one command

$$[R, S] = rga(G) \tag{12.52}$$

Our discussion above was limited to the steady state. Matlab subroutines are available to calculate frequency-dependent RGA, for any frequency or for a sweep of frequencies and can therefore be a useful pre-design checking tool and an aid in pairing, i.e. choosing m SISO control loops $u_i \Rightarrow y_j$*: i, j* $\in [1, \ldots, m]$ *with which to control an* $m \times m$ *process where there is thought to be a danger of destabilising interactions between variables at frequencies other than* $\omega = 0$.

See the reference Monshizadeh-Naini (2009).

12.6.3 Singular value decomposition

The second design approach we discuss uses **singular value decomposition (SVD).**

Controlling n-dimensional multivariable processes by diagonalisation (or similar techniques), followed by the design of single loops:

Straightforward diagonalisation can be applied to 'square' systems with n inputs and n outputs and with n real distinct eigenvalues.

For example, let

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \tag{12.53}$$

with modal matrix

$$E = \begin{bmatrix} 1 & 1 \\ -4 & -1 \end{bmatrix}, E^{-1} = \frac{1}{3}\begin{bmatrix} -1 & -1 \\ 4 & 1 \end{bmatrix} \tag{12.54}$$

Matrix A can be diagonalised to produce the equivalent matrix A' given by

$$\begin{aligned} A' &= \frac{1}{3}\begin{bmatrix} -1 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -4 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned} \tag{12.55}$$

The idea is as follows: If we have an n-dimensional linear process with n actuators and n sensors and if the process model has real distinct eigenvalues, then the square process matrix A may be diagonalised to yield

$$\Lambda = E^{-1}AE \tag{12.56}$$

where Λ is a diagonal matrix of eigenvalues and E is the modal matrix of A with eigenvectors as its columns.

If now, the process is preceded by $E^{-1}$and followed by E, then the combination becomes diagonal with the possibility of designing n non-interacting individual control loops. Note that the variables now being controlled will no longer be the original process-meaningful ones – but instead linear combinations of them.

SVD decomposition works in much the same way as just described but without the severe limitations of squareness and realness.

Using the Scilab or similar command $[U, G, V] = \text{svd}(A)$ will decompose any rectangular matrix to yield three matrices satisfying the relation

$$A = UGV \tag{12.57}$$

where G is a rectangular matrix with the singular values of A arranged (traditionally in descending order of magnitude) on its initial leading diagonal, with the matrices U and V playing the role of the E and E^{-1} matrices discussed above.

Note that again, using this approach, the variables being controlled will no longer be the original process-meaningful ones – but instead, linear combinations of them.

Note: Source material and suggestions for further reading to support the topics of this chapter will be found in Chapter 18.

Chapter 13

Optimisation

13.1 Initial discussion

Optimisation is concerned with finding the best possible solution, formally referred to as the optimal solution, to a particular problem. The term optimisation is often used very loosely in general speech, but in control theory it has a precise meaning: the action of finding the best possible solution as defined by an unambiguous criterion (or cost function).

Optimisation has, to some extent deservedly, acquired a reputation for being out of touch with reality. This is because the analytic techniques for optimisation are highly involved, and in order to make headway, many workers have resorted to drastic modification of the original problem to allow application of some particular optimisation technique; i.e. simplistic assumptions about the problem have, unsurprisingly, produced simplistic solutions. Currently, more healthy attitudes are beginning to prevail. For instance, it is becoming accepted that, for large complex problems, it may be better to encode optimality criteria in more vague but more realistic terms that parallel human evaluation criteria, than to force unwilling problems into an ill-fitting straitjacket to allow rigorous optimisation. With these reservations having been made, it is possible to turn to the ideas and techniques of optimisation theory and practice.

13.2 Optimisation: a few ideas that can form building blocks

Case 1: A mathematical function may take on a maximum value (Figure 13.1).

(a) If we know the 'formula' for the function f, the maximum value can be found by the methods of elementary calculus.

(b) If f is not known as a function, but nevertheless particular values, $f(x_1)$, $f(x_2)$, ..., can be generated for chosen values x_1, x_2, ..., then it will clearly be possible to find the maximum value, to any desired value of accuracy, by numerical search. The efficiency of such a numerical search will vary widely according to the approach used, but almost any conceivable approach would succeed in approaching the maximum to whatever accuracy is required.

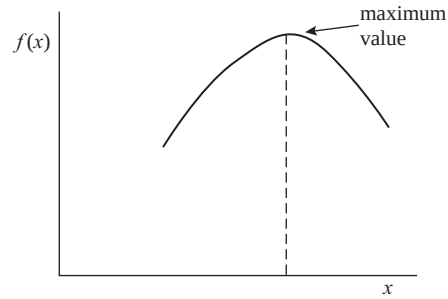


Figure 13.1 The function takes on a maximum value where the first derivative df/dx is zero

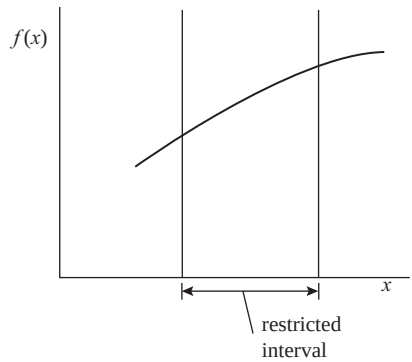


Figure 13.2 The function takes on a maximum value at the upper end of the restricted (closed) interval. Notice that here df/dx ≠ 0

Case 2: A mathematical function, on a restricted interval of x, will always take on a maximum value (Figure 13.2).

Strictly, any continuous function defined on a closed interval will take on maximum and minimum values on that interval.

This is Weierstrass' theorem (see Hardy (1963)).

Note in this case that, as suggested in Figure 13.2, the maximum value may be at a boundary point and that, at a boundary point, the derivative of f will not necessarily be zero and that therefore the ordinary methods of calculus will not suffice to find such maxima.

Case 3: A scalar-valued function of n variables, i.e. $f: \mathbb{R}_n \rightarrow \mathbb{R}_1$, may take on a maximum value.

A scalar-valued function for $n = 2$ is illustrated in Figure 13.3.

(a) If the formula for f is known, then, again, ordinary methods of calculus will suffice to determine the maximum (i.e. $\nabla f = 0$ at the maximum).
(b) If the formula for f is not known but nevertheless particular solutions can be generated numerically, then it is possible to imagine searching in the parameter

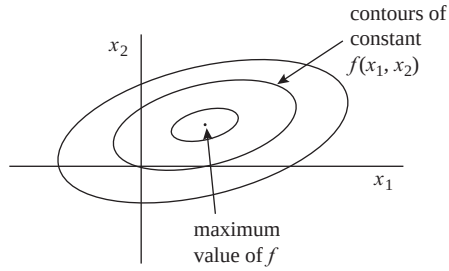


Figure 13.3 The scalar value functions of two variables take on a maximum value where $\partial f/\partial x_1$, $\partial f/\partial x_2$ are both simultaneously zero

space to find the particular values of $x_1, x_2, \ldots, x_n$ that maximise the function, but, whereas in case 1(b) it was clear that any algorithm, however amateur, would eventually locate the maximum value when x was a single variable, it is now by no means obvious how to search n-dimensional parameter space in a meaningful way. Even in the simple case sketched in Figure 13.3 for $n = 2$, considerable ingenuity has to be exercised in devising search algorithms.

Should the function f have a less circular shape in parameter space (i.e. as in Figure 13.4), then successful searching can be expected to be increasingly difficult.

Case 4: A scalar-valued function f: $\mathbb{R}_n \rightarrow \mathbb{R}_1$ defined on a closed region of parameter space, will take on its maximum value on that region (Figure 13.5).

Case 5: One particular function among a set of continuous functions on an interval may maximise a scalar-valued cost function (Figure 13.6).

A specimen problem is as follows: From the set of all continuous real-valued differentiable functions, $u(t) : [t_0, t_{0f}] \rightarrow u(t) \in \mathbb{R}^2 \times t$, choose that particular function $u^*(t)$, $t \in [t_0, t_{0f}]$, that maximises

$$f(u(t)), f : \mathbb{R}^1 \times t \rightarrow \mathbb{R}^1$$

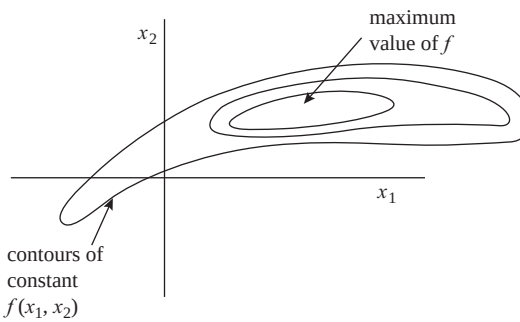


Figure 13.4 Another scalar-valued function of two variables. Here the elongated contours make numerical searching for the maximum difficult

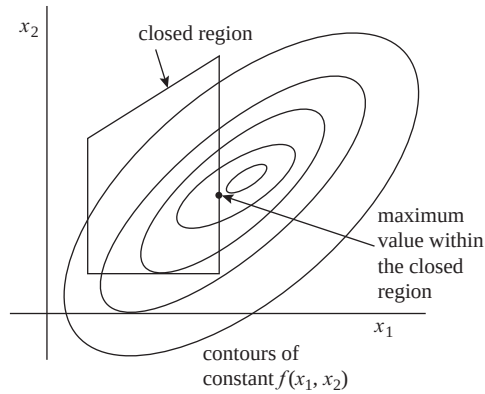


Figure 13.5 A scalar-valued function of two variables will take on its maximum value within the closed region shown. If the maximum is on the boundary of the region, $\partial f/\partial x_1$ and $\partial f/\partial x_2$ will not usually be zero there

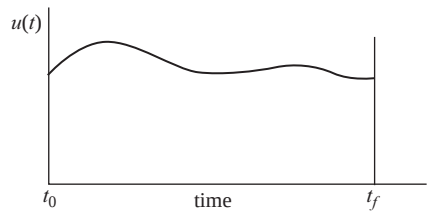


Figure 13.6 A specimen continuous function defined on $[t_0, t_f]$

f is a scalar-valued criterion (cost-function) operating on the set of all real-valued continuous functions $u(t)$ that are defined on the interval $[t_0, t_f]$. Even a casual inspection will show that this problem is very much more difficult than those defined earlier as cases 1 to 4.

An infinite set of candidate functions $u(t)$ exists, and although it is quite easy to envisage finding a numerical approximation to $u(t)$ using some form of computational search algorithm, the analytic method of determining $u(t)$ exactly is a classical mathematical method of great power and beauty.

This analytic method forms part of the subject usually called the calculus of variations, a subject that traditionally was founded by Queen Dido when she maximised the area of land available for founding the city of Carthage about 850 BC.

In its simplest form, the method determines the curve $u(t)$ that, passing through two fixed end points, minimises a given integral.

$$J = \int_0^{t_f} f(u, \dot{u}, t)dt \tag{13.1}$$

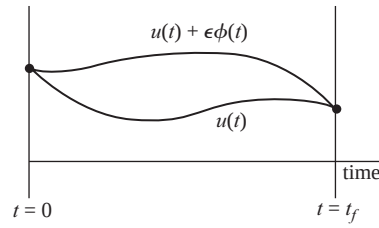


Figure 13.7 The supposed optimal curve $u(t)$ and an arbitrary variant $u(t) + \varepsilon\phi(t)$

Figure 13.7 shows the supposed optimal curve $u(t)$ and one arbitrary variant $u(t) + \varepsilon\phi(t)$, $\phi(t)$ an arbitrary function and ε a scalar. The variant function is approximated by the first two terms of a Taylor series. Manipulation then produces the condition for optimality.

$$J = \int_0^{t_f} \phi\left(\frac{\partial f}{\partial u} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{u}}\right)\right) dt = 0 \tag{13.2}$$

However, $\phi(t)$ was chosen arbitrarily; hence, the optimality condition reduces to

$$\frac{\partial f}{\partial u} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{u}}\right) = 0 \tag{13.3}$$

This is the Euler–Lagrange necessary condition for optimality of the curve.

Use of the calculus of variations to solve control problems: In optimal control problems, the differential equations that model the process to be controlled must be satisfied at all times, while, simultaneously, the Euler–Lagrange conditions have to be met. The extension of the calculus of variations to meet this requirement is usually performed by the use of Lagrange multipliers.

Suppose that the optimal control problem is to choose $u(t)$ on the interval $[0, t_f]$ so that the process with model

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - u \end{aligned} \tag{13.4}$$

behaves so as to minimise

$$J = \int_0^{t_f} f(x, \dot{x}, t) \tag{13.5}$$

The Lagrange multipliers λ_1, λ_2 are introduced by enhancing the expression for J to

$$J = \int_0^{t_f} f(x, \dot{x}, t) + \lambda_1(\dot{x}_1 - x_2) + \lambda_2(\dot{x}_2 - x_1 + u) \tag{13.6}$$

Minimisation of the enhanced expression for J, still using the calculus of variations approach, will now minimise the original J while satisfying the equality constraints imposed by the process. After the Lagrange multipliers have served their purpose in this way, they are eliminated by substitution. The result obtained is an optimal control strategy, specifying a curve $u_{opt}(t)$ on the interval $[0, t_f]$ that, when input to the process (as modelled by (13.4)), will result in a performance that minimises J.

Case 6: One particular function among a set of functions satisfying an inequality constraint may maximise a scalar-valued cost function.
A specimen problem is as follows: From the set of (not necessarily continuous) functions

$$u(t) : [t_0, t_f] \rightarrow u(t) \times \mathbb{R}^1 \times t \tag{13.7}$$

that satisfy the constraint $||u(t)|| \leq m$ (m is a constant, for all t in $[t_0, t_f]$)

($||x||$indicates the **norm**, being a scalar-valued measure of a vector, matrix or function.) Let us choose that particular function $u^*(t) : [t_0, t_f] \rightarrow$ that maximises

$$f(u(t)), \quad f : \mathbb{R}^1 \times t \rightarrow \mathbb{R}^1 \tag{13.8}$$

Notice that $u(t)$ has to remain within the admissible region shown in Figure 13.8.

We observe that many practical optimisation problems arising in control applications are subject to a constraint on signal magnitude similar to (or possibly more complex than) the constraint outlined here. Very often, the optimal function $u^*(t)$ will be found to take values on the boundary of the admissible region for some or all of the time period (t_0, t_f), as in Figure 13.9.

This problem may turn out to be either more or less difficult than that of case 5.

It is more difficult than case 5 in that the presence of the constraint makes it more difficult to apply methods analogous to ordinary calculus. The problem may be easier than that of case 5 in those cases where it is possible to say in advance that the optimal solution $u^*(t)$ operates along the boundaries of the region during the whole of the time period (t_0, t_f) with a finite number of switchings between these extreme values. Finding the optimal solution $u^*(t)$ then amounts to the simpler (?) problem of determining the finite set of switchover times.

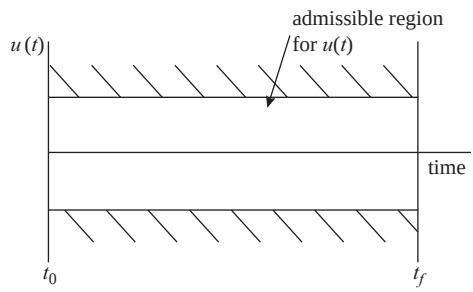


Figure 13.8 The admissible region in which u(t) must remain

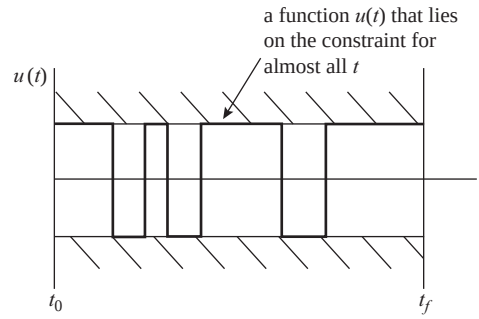


Figure 13.9 The optimal $u^*(t)$ will often take values on the boundary of the admissible region

The above six problems, cases 1 to 6, illustrate in a simplified way the range and nature of optimisation problems that are encountered in control theory. It must be emphasised though that the problems 1 to 6 as described concentrate only on the core features. Any realistic optimisation problem requires a quite extensive framework involving dynamic system models, a possibly complex criterion function and, where appropriate, mechanisms for taking constraints into account.

13.2.1 Discussion

We begin by listing some general points:

(i) Even the simplest optimal control problem involves a process model and a cost function J. The process model can be considered to impose equality constants on the minimisation of J.

(ii) The choice of J is difficult in every real case – a compromise always has to be reached between relevance and mathematical tractability. Forcing a complex, often unquantifiable, problem to have a simplistic cost function is likely to lead to results that have little practical value.

(iii) In most control problems, the magnitude of the controls must not exceed certain upper limits. The upper limits can be considered to be inequality constraints on the minimisation of J.

(iv) Inequality constraints (see iii) prevent the calculus of variations being applied. Pontryagin's maximum principle or the method of dynamic programming then need to be used. (The situation to be dealt with is essentially a generalisation of that where a function defined on a closed interval is to be maximised – the methods of ordinary calculus cannot be used because the maximum may not be a turning point – see Figure 13.2.)

(v) The methods discussed above all yield open loop optimisation strategies, i.e. they specify $u_{opt}(t)$ for all t in the time interval of interest. It is usually impractical to implement open loop optimisation, except in a few special cases, and the strategies need to be converted to closed loop algorithms. This conversion is always possible provided that J is a quadratic form and that the

process model is linear. Under these, very restrictive, conditions the optimal feedback law is yielded by the solution of a Riccati equation. Even then, the Riccati equation has time-varying coefficients, making it difficult to implement, unless the optimisation horizon is infinite.

(vi) The optimal feedback algorithm produced by the solution of the Riccati equation usually requires all of the process states to be measurable online. If some of the process states are inaccessible, a state estimator will need to be developed to make those states available online.

(vii) If a state estimator feeds an optimal feedback algorithm, the question arises: Does the combination of optimal estimator and optimal controller yield the overall optimum solution (since usually, a set of interconnected optimal subsystems would not combine into an overall optimal system)? This problem is addressed by the separation theorem. This roughly states that, if the system is linear, the noise signals Gaussian and the cost function quadratic, then overall optimisation will be yielded by a combination of optimal state estimator and optimal feedback controller.

(viii) The effects discussed in (i)–(vii) add together to make realistic optimisation of a real process a very difficult task indeed. There is nevertheless a great potential for optimisation techniques to lead the way in approaches to the co-ordination of complex processes involving many hundreds of elements and in extracting additional productivity from systems containing complex process mechanisms (such as micro-organisms).

(ix) The literature on optimisation is enormous. Some suggestions are made in Section 18.9.

(a) We now go on to discuss one particular optimisation problem – that of time-optimal control. This topic forms just one aspect of optimisation as discussed above and in no sense is it different or isolated. Here, it has been singled out for amplification because the development is quite pleasing, leading to a geometric interpretation and a link across to operator methods of system representation.

13.3 Time-optimal control

Assume that, in the system of Figure 13.10, the shaft is at rest at position θ_0 and it is required to bring it in minimum time to a new rest position θ_1.

We can think of the problem in the following way: the quantity $(\theta_1 - \theta_0)$ is fixed and all possible solutions can be sketched as velocity–time graphs. It is clear that, to obtain a minimum time solution, we must have the steepest initial rise in velocity followed by the steepest possible fall (since, in graphs like that of Figure 13.11, we need to generate maximum area beneath the graph in the shortest time interval; i.e. the ideal velocity profile is rectangular with infinite acceleration/deceleration).

Idealised situations in which there are constraints on velocity but not on acceleration (case *a*) and vice versa (case *b*) are shown in Figure 13.12. It can be seen that the minimum time solution is only meaningful if there are constraints on

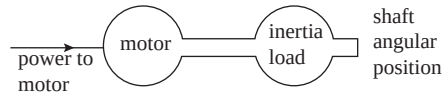


Figure 13.10 A motor driving an inertia load

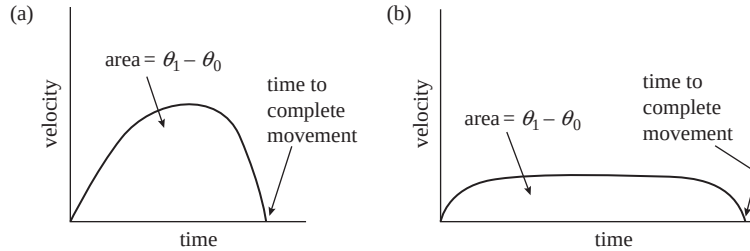


Figure 13.11 Two possible velocity profiles that each result in the movement of the shaft from position θ_0 to position θ_1

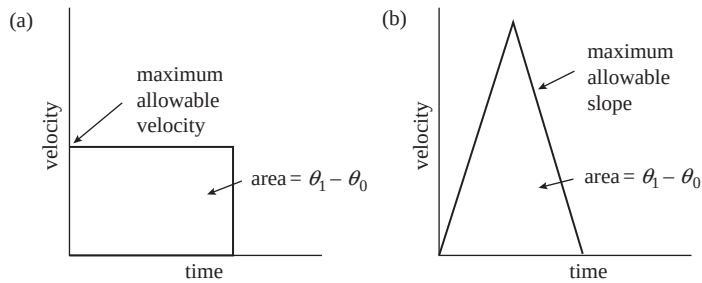


Figure 13.12 (a) Minimum time velocity profile for the case where velocity is constrained but acceleration is unconstrained; (b) minimum time velocity profile for the case where acceleration is constrained but velocity is unconstrained

velocity or acceleration – for otherwise the minimum time would approach zero as the acceleration/deceleration increased without limit (Figure 13.13).

Thus, we can see that the minimum time solution requires maximum acceleration followed by maximum deceleration, with the only decision being the time at which the changeover is to be made between these regimes. A control that stays on one constraint or another all the time (rigorously – almost all the time) is called a bang-bang control.

It is a result in optimal control theory that every minimum-time control problem has a bang-bang solution and it therefore follows that if the minimum-time control problem has a unique solution then that solution is a bang-bang solution.

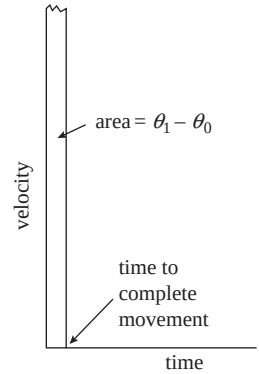


Figure 13.13 A minimum time solution with no imposed constraints tends in the limit to an infinite velocity spike

13A Time-optimal control: a geometric view

Let the system of Figure 13.14 be at an initial state x_0 at time t_0. Consider a time $t_1 > t_0$ and let Ω represent the set of admissible (i.e. constrained) controls defined on the closed interval (t_0, t_1).

Now let $\mathcal{R}_1$ represent the region in state space X to which the state x can be driven in time $t_1 - t_0$ by the application of all possible admissible controls in Ω_1. Consider next a *t*ime $t_2 > t_1$, and let Ω_2 represent the set of admissible controls defined on the interval $[t_0, t_2]$. It is clear that the region $\mathcal{R}_2$ in X to which the state can be driven in time $t_2 - t_0$ must contain the region $\mathcal{R}_1$.

Thus, considering times $t_1, t_2, \ldots, t_n$, with $t_n > \ldots t_2 > t_1$, the reachable regions in state space will have the form shown in Figure 13.15. The meaning of these regions is that any point x in region $\mathcal{R}_i$ can be reached in time $t_i - t_0$. Under reasonable assumptions of smoothness, the region $\mathcal{R}$ grows smoothly with increasing time, so that, given any chosen point x_d, there exists some unique time, say t^*, for which $x_d \in \delta\mathcal{R}(t^*)$, i.e. x_d is a boundary point of the closed set $\mathcal{R}$ (t^*).

This means that:

(i) x_d cannot be reached from x_0 by the application of admissible controls in any time $t < t^*$.
(ii) x_d can be reached in time t^* and, because (i) applies, t^* can be seen to be the minimum time.

To summarise, a point x_d can be reached in minimum time t^* if and only if x_d belongs to the boundary $\delta\mathcal{R}(t*)$ of the reachable set $\mathcal{R}(t^*)$ (see Figure 13.16).

In Figure 13.15, x_d cannot be reached in time t_1. x_d can be reached in time t^* and this will be the minimum time solution. x_d can be reached in time t_2 but this is not the minimum possible time. In this case, if the requirement is to reach point x_d at time t_2, the problem is not a minimum time problem.

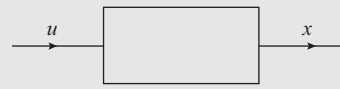


Figure 13.14 The system whose time-optimal control we study

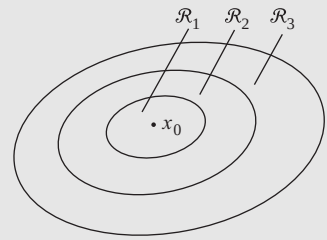


Figure 13.15 Reachable regions in state space

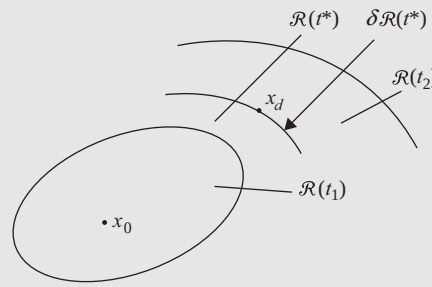


Figure 13.16 x_d belongs to the boundary of the region $\mathcal{R}(t^*)$

The shape of the reachable set $\mathcal{R}$

We have already observed the useful property that the set $\mathcal{R}$ grows smoothly with time. Now we turn to examine the shape of $\mathcal{R}$. If the system that we are studying is linear then it can be represented by a linear transformation, say P, operating on the initial condition x_0 and the chosen control $u(t)$ to produce a new state, i.e.

$$P : (x_0, u) \rightarrow x \tag{13.9}$$

where $x_0 \in X$, $u \in \Omega(t_0, t_f)$ for some fixed t_f,

$$x \in X \tag{13.10}$$

And in this sense we can define $\mathcal{R}(t_f)$ as

$$\mathcal{R}(t_f) = \{x | u \in \Omega\} \tag{13.11}$$

This can be stated more simply as

$$P : \Omega \rightarrow \mathcal{R} \tag{13.12}$$

i.e. the linear transformation P maps the region Ω into the region $\mathcal{R}$.

We now note that convexity is invariant under linear transformation (see Hardy (1963)), and thus if the set Ω is convex (strictly convex), then $\mathcal{R}$ will also be convex (strictly convex), provided that the system under study is linear (Figure 13.17).

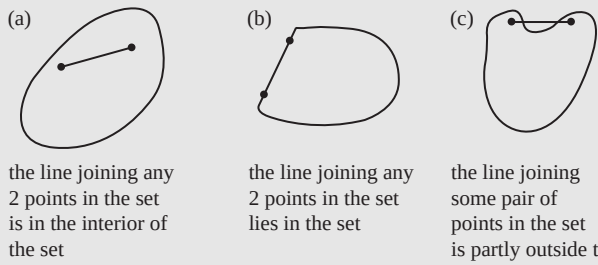


Figure 13.17 (a) A strictly convex set; (b) a convex set; (c) a non-convex set

Geometrically, a set C is convex if the line joining any two points in C belongs wholly to C. (For strict convexity, the line joining every two points must be in the interior of C.)

The shape of the set Ω of admissible controls

It is surprisingly rewarding to examine the shape of Ω as it relates to practical constrained control problems. For simplicity, we will concentrate on the case where the control input u is a vector with two elements $u_1(t)$, $u_2(t)$.

The most common constraints encountered in practical applications are as follows:

(a) $u_1(t)_2 + u_2(t)_2 \leq m$ for all t, m a fixed scalar
(b) $|u_1(t)| + |u_2(t)| \leq m$ for all t, m a fixed scalar
(c) $\max\{|u_1(t)|, |u_2(t)|\} \leq m$ for all t, m a fixed scalar

The shape of these constrained sets for the three cases is shown in Figure 13.18. (This is the usual Euclidian norm on the space U.)

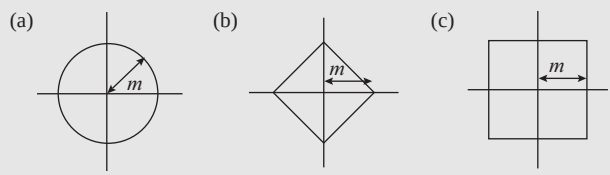


Figure 13.18 (a) The set Ω for the Euclidean norm; (b) the set Ω for the absolute value norm; (c) the set Ω for the maximum value norm

All the sets are convex but only case A has a strictly convex constraint set. (Almost all comparisons seem to end up confirming the superiority of least squares as a criterion.)

Thus, for linear systems with constraints on U defined by approaches (a), (b) and (c), the set $\mathcal{R}$ will have one of the shapes sketched above.

The significance of the shape of the set $\mathcal{R}$

It can also be shown that, if the set $\mathcal{R}$ is compact, then the optimal control u to reach x_d is unique if $x_d \in \mathcal{R}(t)$ for some t. The interior mapping theorem then shows that u must attain its maximum if it is to be an optimal control. Finally, Lyapunov's theorem that the range of a vector measure is closed allows the bang-bang nature of a unique time-optimal control to be proved. Geometrically, this shows that time optimality requires $x_d \in \mathcal{R}(t^*)$ and that the pre-image of x_d in Ω belongs to $\delta\Omega$ (Figure 13.19).

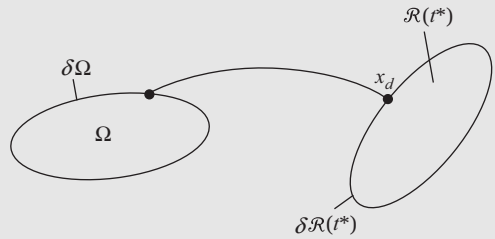


Figure 13.19 Time optimality requires that the pre-image of $\mathrm{x_d}$ *belongs to* $\delta\Omega$

13B The following geometric argument can form a basis for the development of algorithms or for the proof of the Pontryagin maximum principle

Let $x_d \in \mathcal{R}(t^*)$; then there exists a hyperplane M that supports $\mathcal{R}$ at x_d. M can be represented as the translation of the null space of some non-linear functional g on the space X, i.e.

$$M = \{x | g(x) = C\}, C \text{ a real number} \tag{13.13}$$

can also (Riez representation theorem) be written

$$\langle x, g \rangle \tag{13.14}$$

where g is normal to the hyperplane M

$$x_d \in \mathcal{R}(t^*) \cap M \tag{13.15}$$

and

$$\langle x, g \rangle = \sup_x \langle x, g \rangle \tag{13.16}$$

i.e. x_d is the farthest point from x in the set $\mathcal{R}(t^*)$ in the direction g but $x = Pu$, while if $x_d = Pu$ with $x_d \in \mathcal{R}(t)$, then u is an optimal control on [0, t^*]. Further, if x_d is an extreme point of $\mathcal{R}(t^*)$, then u is the unique optimal control.

13C Construction of time-optimal controls

A linear dynamic system has equations

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{13.17}$$

or equivalently

$$\begin{aligned} x(t) &= \Phi(t)x(t_0)\int_{t_0}^{t}\Phi(t-\tau)B(\tau)u(\tau)d\tau \\ &= \Phi(t)x(t_0) + \Phi(t)\int_{t_0}^{t}\Phi(-\tau)B(\tau)u(\tau)d\tau \end{aligned} \tag{13.18}$$

The control problem is, given $x_d \in X$, choose $u(t) \in \Omega$ where $\Omega = \{u||u(t)| \leq k\}$ such that

(i) $x(t^*) = x_d$
(ii) $t^* = \inf\{tx(t) = x_d,\ t \geq t_0\}$

and define

$$e(t) = \Phi^{-1}(t)(x_d - x(t_0)) \tag{13.19}$$

The control objective then is to choose u such that

$$e(t^*) = \int_{t_0}^{t^*}\Phi^{-1}(\tau)B(\tau)u(\tau)d\tau \triangleq \int_{t_0}^{t^*}Q(\tau)u(\tau)d(\tau) \tag{13.20}$$

Assume there exists an optimal control, then, necessarily,

$$e(t^*) \in \delta A(t^*) \cap M \tag{13.21}$$

where

$$\mathcal{A}(t^*) \triangleq \{r(t^*)|u \in \Omega \times t_0 t^*]\} \tag{13.22}$$

$\mathcal{A}$ is called the attainable set

$$M = \{x|g(x) = C\}, r(t) = \int_{t_0}^{t}Q(\tau)u(\tau)d\tau \tag{13.23}$$

(M is a hyperplane, g is a functional on X) for some function g and for some constant C.

Now, as we have seen, for optimality,

$$\begin{aligned} \langle e(t^*), g\rangle &= \sup_u\langle r(t^*), g\rangle = \sup_u\int_{t_0}^{t^*}Q(\tau)u(\tau)gd\tau \\ &\leq \left(\int_{t_0}^{t^*}|Q(\tau)g|^q d\tau\right)^{1/q}||u||_p \leq k\left(\int_{t_0}^{t^*}|Q(\tau)g|^q d\tau\right)^{1/q} \end{aligned} \tag{13.24}$$

The condition for optimality is that equality should exist in the inequality chain, i.e.

$$u(\tau) = a|Q(\tau)g|^{q/p}\text{sign}(Q(\tau)g) \tag{13.25}$$

where α is a constant to be determined and

$$\frac{1}{q} + \frac{1}{p} = 1 \tag{13.26}$$

When U is an L^{∞} space, the optimality condition reduces to

$$u(\tau) = \alpha\,\text{sign}(Q(\tau)g) \tag{13.27}$$

but $\|u\|_{\infty} = k$ for optimality; hence, $\alpha = k$ to give

$$u(\tau) = k\,\text{sign}(Q(\tau)g) \tag{13.28}$$

t^* and g have to be computed, and Kranc and Sarachik (1963) suggest appropriate methods.

13D The (matrix) Riccati equation: some history

The matrix Riccati equation is most often encountered at an intermediate stage in the numerical calculation of quadratic optimal control/optimal observation algorithms. The equation looks quite forbidding but is now easily solved numerically by a Linear Quadratic Regulator (LQR) algorithm (for instance) but one can understand how, before about 1975 and the arrival of easily available high-level software tools such as MATLAB, the Riccati equation was too intractable to be solved without significant effort. Other methods were therefore used to solve optimisation problems and Riccati equations were rarely mentioned except in specialist mathematical journal papers.

This observation was confirmed by a point sampling of a few mainstream control references from the past – e.g. Truxal (1955), Zadeh (1963) and Beveridge (1970) contain no references to LQ or Riccati and it is not until Sage and White (1977) (Chapter 5.1) that the first in-depth coverage can be found in the mainstream literature.

Therefore, it may come as a surprise that Count Jacopo Riccati published his equation in 1724 and that Daniel Bernoulli's earliest mathematical publication in 1725 offered a solution to it and that it was the centre of considerable intellectual activity before descending into relative obscurity. The pedigree of the equation is complete once it is realised that the name was bestowed on it by none other than Leonard Euler in 1733.

It would surely be interesting to have samples of the applications that might have motivated Riccati to publish examples of his differential equation in, for instance, Riccati (1724) and which drew a response from Daniel Bernoulli. According to Bittanti (1996), one such motivation was the following problem

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \tag{13.29}$$

which describes the trajectory of a point in the (α, β) plane.

Riccati was interested in the behaviour of $x = \frac{\beta}{\alpha}$, which is governed by the eponymous Riccati equation $\dot{x} = ax^2 + bx + x$, where $a = -w_{12}$, $b = -w_{22} - w_{11}$, $c = w_{21}$.

Some interesting references:

Bernoulli D. 'Solutio problematis Riccatiani propositi'. *ASct. Lips. Suppl. Tom. VIII,* p.73. *ActaEruditorum*. 1725:473–75

Bittanti S., Laub A., Willems J. The Riccati Equation (Berlin, Springer-Verlag, 1991)

Bittanti S. 'History and prehistory of the Riccati equation'. *Session Origins of the calculus of variations and optimal control, IEEE Conference on Decision and Control, Kobe, Japan;* December 1996 (300 years after the publication of Bernoulli's paper on the 'brachystochrone problem' in the *ActaEroditorumLipsae* (1696))

Riccati J.F. Animadversiones In Aequationes Differentiates, Acta Eruditorum Lipsiae. (1724). Re-printed by Bittanti S. (ed.). *Count Riccati and the Early Days of the Riccati Equation* (Bologna, Milano, Pitagora Editrice, 1989)

13.4 Linear quadratic optimisation

13.4.1 LQR problems with infinite time horizon

LQR problems are Linear Quadratic Regulator problems, meaning that they deal with linear dynamic models, that the usually unique solutions are those that minimise a quadratic (i.e. least squares) criterion and that the assumed underlying problem is regulation (i.e. to control some process so as to minimise the effects of disturbances).

The relevance of the infinite time horizon is that, in such cases, the optimal control strategy is always time invariant and, thus, one is seeking a steady state strategy.

Given the standard continuous time state equation

$$x' = Ax + Bu \tag{13.30}$$

it is assumed that all elements of the state vector x are accessible to be fed back directly through an appropriately dimensioned gain matrix K, so that $u = -Kx$ and the state equation becomes

$$\begin{aligned} x' &= Ax + BKx \\ &= (A + BK)x \end{aligned} \tag{13.31}$$

Notice that although the control signal is no longer visible in the above equation, it is nevertheless very active and in any proposed design its magnitude and rate of change need to be kept under review to ensure that the inevitable linearity assumptions are not too drastically violated.

It is a useful viewpoint to consider the matrix $(A + BK)$ to be the new system matrix that we, as system designers, can modify through choice of the feedback gain matrix K.

Clearly, we might attempt to specify and then achieve pole locations for the matrix $(A + BK)$. However, it is not easy, in a multivariable situation to decide on the pole locations that will achieve a desired real-world performance and the LQR approach uses a more understandable scalar-valued time domain cost function

$$J = \int_0^\infty (x^{\mathrm{T}} Q x + u^{\mathrm{T}} R u) dt \tag{13.32}$$

Q, R are (usually chosen to be) diagonal weighting matrices that indicate the customer-specified relative priorities on tight control of the states (Q) and minimisation of the costs of control actions (R). To guarantee the existence of a solution, it is necessary that Q is positive semi-definite and that R is positive definite. J can be calculated and minimised by an LQR optimisation routine that operates by solving the continuous time algebraic Riccati equation

$$A^T X + XA - XBR^{-1}B^T X + Q = 0 \tag{13.33}$$

where X is an $n \times n$ symmetric matrix to be determined and A, B, Q, R are given real matrices that specify the problem.

Once we are in possession of the matrix X, we are able to calculate the optimal control signal u_{opt} that will minimise J from the equation

$$u_{opt} = -R^{-1}B^T X x = Kx \tag{13.34}$$

so that

$$(Ax + Bu_{opt}) = (A + BK)x \tag{13.35}$$

We proceed to a simple example with an open loop stable process that we shall solve for widely different choices of Q and R and then shall repeat the procedure for an unstable process. Some of the benefits of the design approach will hopefully become apparent.

13.4.1.1 Solution of the LQR problem for three different choices of cost function: open loop stable process (see Figure 13.20, where the dynamic performances are compared)

The process has two states and one input and it is assumed that both states are available to be fed back.

Our process can be thought of as a motor with inertia and friction having angular position x_1 and angular velocity x_2 (so that $x_2 = x_1'$).

In what follows, we use for conciseness, inline representations of matrices, so we have A (2×2) and B (2×1):

$A = [0, 1; -2, -3]$; $B = [0.85; 0.68]$; //two states, one input

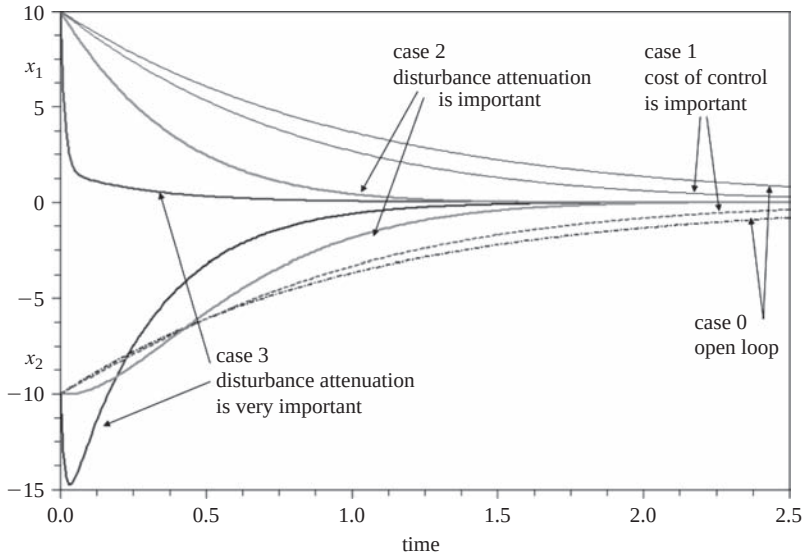


Figure 13.20 Simulated time responses of the four systems, cases 0–4, given an initial condition of x = [10; –10]

Case 0: For comparison
This is a stable open loop system with real poles at $s = -1$, $s = -2$ and will have reasonable disturbance attenuation properties even without feedback. The transition matrix $\Phi(t)$, $t = 1$ for this open loop system is

$$[0.600, 0.233; -0.465, -0.097] \tag{13.36}$$

(In the comparisons that follow, the transition matrices will be quoted for $t = 1$ in all cases, in the belief that, perhaps after a little practice, they allow a rapid view of a system's transient response.)

Case 1: Cost of control is important
$Q = \text{diag}[1, 2.5]$; $R = 4$
$X = [1.466, 0.1378; 0.1378, 0.455]$
$K = [–0.335, –0.107]$
$A + BK = [–0.285, 0.909; –2.227, –3.072]$
Eigenvalues of $A + BK = –1.679 \pm j0.288$
Transition matrix of $A + BK = [0.435, 0.167; –0.410, –0.078]$

Case 2: Disturbance attenuation is important
$Q = \text{diag}[4, 10]$; $R = 1$ needs redoing with $R = 1$ not 4
$X = [3.149, –0.425; –0.425, 1.459]$
$K = [–2.39, –0.631]$
$A + BK = [–2.029, 0.463; –3.623, –3.429]$

Eigenvalues of $A + BK = -2.729 \pm j1.09$
Transition matrix of $A + BK = [0.067, 0.025; -0.192, -0.007]$

Case 3: Disturbance attenuation is very important; cost of control negligible $Q = diag([400,400]);\ R = 0.1$
$X = [41.393, -40.709; -40.709, 52.051]$
$K = [-75.02, -7.92]$
$A + BK = [-63.77, -5.73; -53.01, -8.39]$
Eigenvalues of $A + BK = -68.80, -3.36$
Transition matrix of $A + BK = [0.00268, -0.00306; -0.0283, 0.0322]$

Comments on the results

Case 1: Quite low feedback gains, little change to the original A matrix except that the zero element a_{11} is replaced by a non-zero element in the $A + BK$ matrix. The response of case 1 shows little improvement on the open loop system.

Case 2: Modest feedback gains, nevertheless produce significant changes to the original A matrix and to the performance.

Case 3: Very high feedback gain, drastic changes to the system matrix and an ultra-rapid correction of the initial condition with x_1 being reduced even at the cost of initially sending x_2 off in the wrong direction. (Note though that, in this particular case, $x_2 = x_1'$ so that the two variables are highly dependent.)

It would be valuable to look back at the results of the three sets of calculations above to obtain a quantitative feel for the feedback gains, $A + BK$ matrices, eigenvalues and transition matrices for $t = 1$ that are produced by the basic LQR method.

We now proceed to repeat the above treatment for three similar cost functions, but now on a process that is open loop unstable. What we shall find is that the LQR method, with its guarantee of producing a feedback gain that guarantees to stabilise an unstable process and allows us to view a wide range of possible responses, has quite a lot to offer.

13.4.1.2 Solution of the Riccati equation for three different choices of cost function: open loop unstable process

Case 0

We have reversed the signs of elements a_{21} and a_{22} in our previous open loop stable example to obtain

$A = [0, 1; 2, 3]$
as before $B = [0.85; 0.68]$

Eigenvalues of the open loop system are $+3.56$ and -0.56, confirming the expected instability and the transition matrix

$\Phi(1) = [5.28, 8.40; 16.8, 30.5]$ confirms the rapid exponential growth of the unforced response.

This response will not be plotted.

Case 1

$Q = \text{diag}([1, 2.5])$; $R = 4$

$K = [-3.43, -6.47]$

$A + BK = [-2.91, -4.50; -0.333, -1.400]$

Eigenvalues are –3.60, –0.718.

Case 2

$Q = \text{diag}([4,10])$; $R = 1$

$K = [-3.32, -8.859]$

$A + B^*K = [-2.82, -6.53; -0.259, -3.02]$

Eigenvalues are –4.23 and –1.618749.

Case 3

$Q = \text{diag}([400, 400])$; $R = 0.1$

$K = [-5.93, -101.73]$

$A + BK = [-5.04, -85.47; -2.03, -66.17]$

Eigenvalues are –68.9, –2.32.

The three responses to initial conditions $x = [10; -10]$ are shown in Figure 13.21.

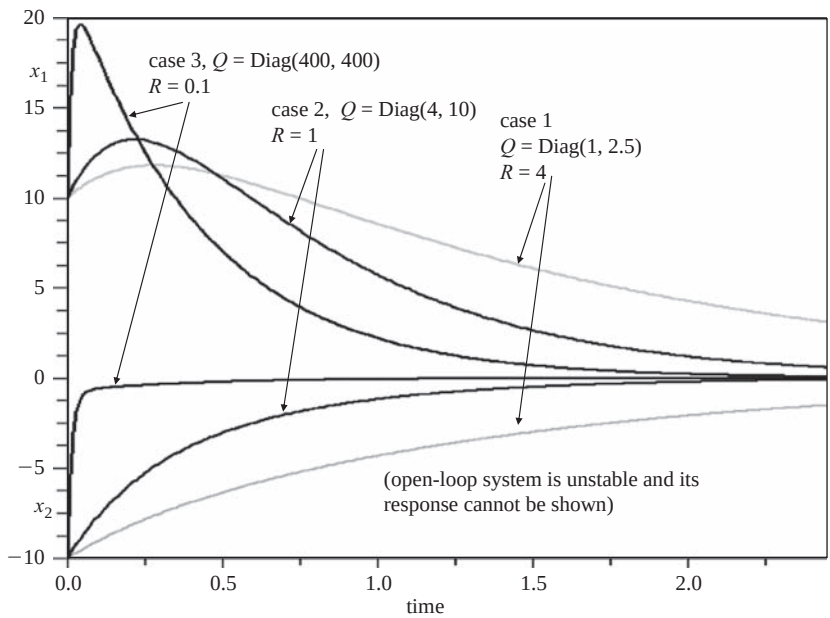


Figure 13.21 Simulated responses of the three LQR calculated state feedback algorithms when applied to the open loop unstable test process $A = [0, 1; 2, 3]$, $B = [0.85; 0.68]$. The effect of the close linking of x_1 and x_2 (one being the integral of the other) is most clearly seen for case 3

We use again the unstable process $A = [0, 1; 2, 3]$; $B = [0.85; 0.68]$ and set $Q = [1, 0; 0, 50]$; $R = 0.01$.

Using the LQR software, we obtain

$X = [0.217, -0.226; -0.226, 1.402]$

$K = [-3.0779182, -76.133013]$

$A + B^*K = [-2.6162305, -63.713061; -0.0929844, -48.770449]$

Eigenvalues are –48.89, –2.48.

Next, the weightings in the Q matrix are drastically changed over to

$Q = [50, 0; 0, 1]$; $R = 0.01$

Now we obtain,

$X = [9.979, -12.36; -12.36, 16.748016]$

$K = [-7.274, -87.61]$

$(A + B^*K) = [-6.18, -73.46; -2.95, -56.57]$

Eigenvalues are –60.55, –2.21.

The responses to initial condition $x_0 = [10; -10]$ are shown in Figure 13.22 below. It can be seen that the strong dynamic linking between the two state variables has prevented the widely differing priorities expressed in the two Q matrices from being achieved: the two sets of responses are the same for practical purposes.

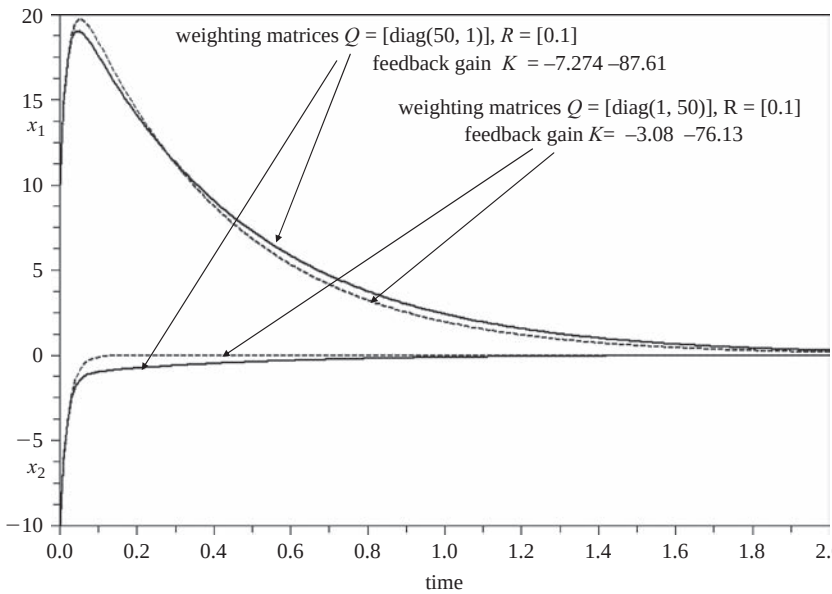


Figure 13.22 Although in this example, two widely differing Q matrices were specified, reflecting widely differing priorities on desired responses, the two sets of responses are almost the same. This example demonstrates that the structure of the dynamic model and its internal links will necessarily limit independence of action

13.4.2 LQR problems with finite time horizon

Next we demonstrate the solution to an ultra-simple but easy-to-appreciate finite-horizon optimisation problem that is posed in discrete time. The same simple problem is solved by hand four times, each time with different weights in the optimisation cost function, allowing the reader to see clearly how the nature of the cost function affects the nature of the solution.

The author would like to thank Professor Guy Beale (George Mason University, VA, USA) for permission to publish the following extracts from his Discrete-Time LQR Example #1

A discrete time state space model will be used

$$x_{k+1} = Ax_k + u_k, x_0 = 100 \tag{13.37}$$

The matrix A will be a scalar, either 0.5 or 2, with the second choice leading to an unstable process that needs to be controlled and, as we shall see, requires more control effort.

The cost index

$$J = 0.5\left\{S_N x_4^2 + \sum_0^3 (Qx_k^2 + Ru_k^2)\right\} \tag{13.38}$$

is used with two values for Q and R and with S_N (always taking the same value) representing a fixed cost for the final value after four time steps. Thus, there are the following four situations to be evaluated (Table 13.1; Figures 13.23–13.27).

Because of the fixed time horizon, the control law will necessarily be time varying and will take the form

$$u_k = -K_k x_x \tag{13.39}$$

where the feedback gain matrix G_k is given by

$$K_k = (B^{\mathrm{T}} X_{k+1} B + R)^{-1} B^{\mathrm{T}} X_{k+1} A \tag{13.40}$$

$$K_k(\text{in this scalar case}) = \frac{AX_{k+1}}{X_{k+1} + R} \tag{13.41}$$

Table 13.1 Parameters for the four cases

Case number	*A*	*Q*	*R*
1	2	20	2
2	2	2	20
3	0.5	20	2
4	0.5	2	20

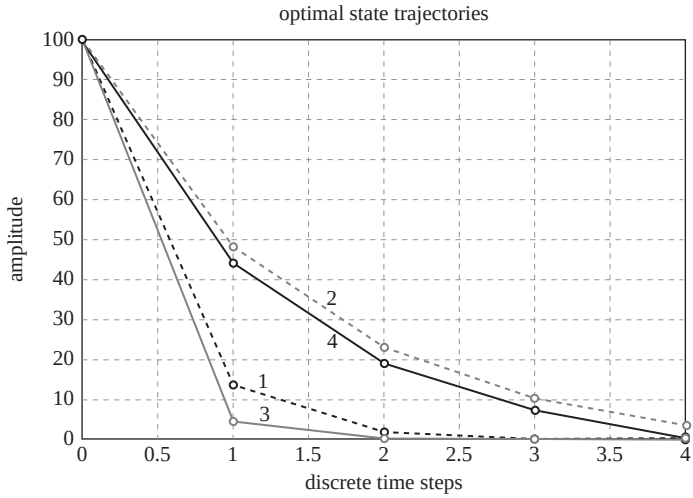


Figure 13.23 Responses of the state variable x for the four cases. Notice that cases 2 and 4, where, by setting $R = 20$, a high cost was put on control actions, are slower to eliminate the initial disturbance than are cases 1 and 3

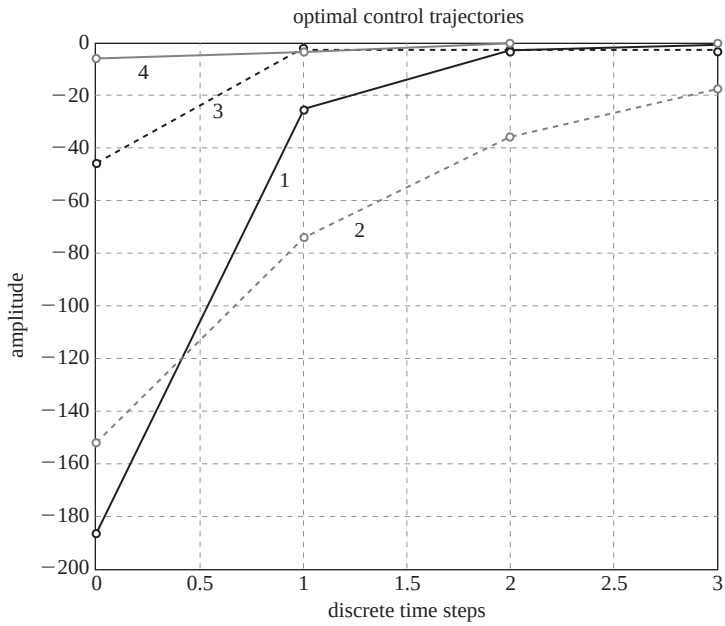


Figure 13.24 Plots of u for the four cases. Notice that cases 1 and 2, where the process is open loop unstable with $A = 2$, require more vigorous control action than the open loop stable cases 3 and 4

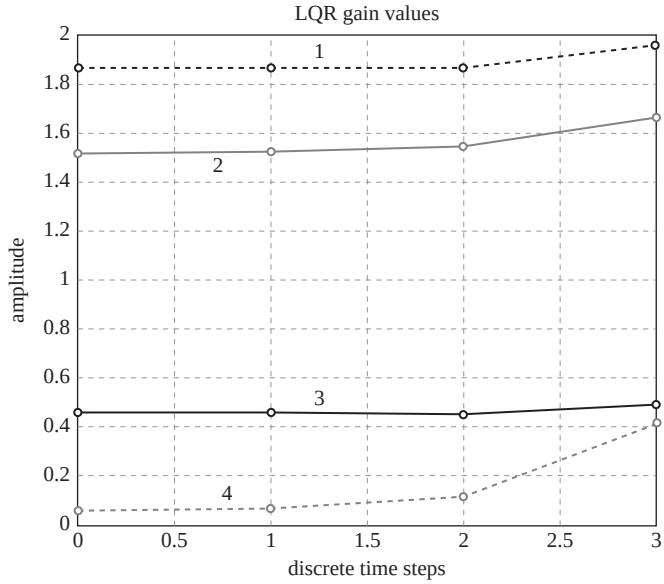


Figure 13.25 The time-varying values of the feedback gains for the four cases. They are ordered from the highest: case 1, unstable process, high weight on state deviation, to the lowest case 4, stable process, corrective action penalised by the cost function

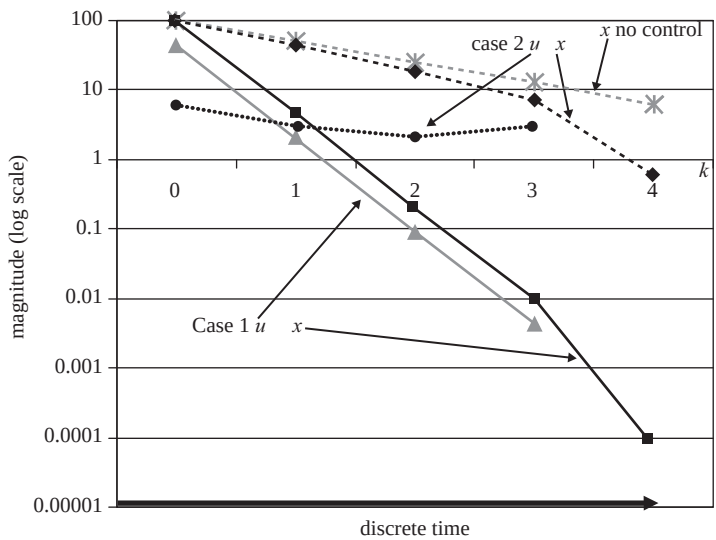


Figure 13.26 Control signals u and responses x for cases 1 and 2, (A = 0.5), together with a plot of the response of x when no control is applied

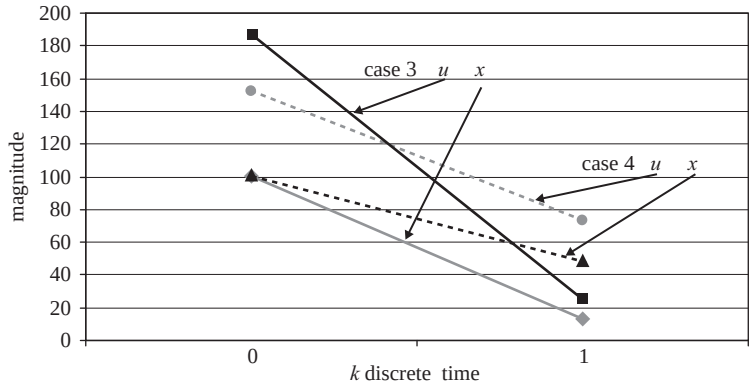


Figure 13.27 Here, for clarity, we show only the results (linear scale) for time steps $k = 0$ and $k = 1$. This graph shows that for case 3, an initial value $u_0 = 186$ quickly drives x towards the origin and after that (see Figure 13.26) quite low values of u suffice to attain the optimal behaviour

and the Riccati equation for X is

$$X_k = A^{\mathrm{T}}(X_{k+1} - X_{k+1}B[B^{\mathrm{T}}X_{k+1}B + R]^{-1}B^{\mathrm{T}}X_{k+1})A + Q$$

$$= (\text{for scalars})\frac{A^2RX_{k+1}}{X_{k+1} + R} \tag{13.42}$$

But $X_4 \triangleq 100$, *from the problem definition*

For case 1 where $A = 2$, $Q = 20$, $R = 2$, we solve the Riccati equation for the discrete times $k = 3$, 2, 1 and 0, in that order, in other words in backwards time, to obtain, for $k = N$–1 to $k = 0$ the following values for X_k:

$$X_3 = \frac{A^2RX_4}{X_4 + R} + Q = \frac{4 \cdot 2 \cdot 100}{100 + 2} + 20 = 27.843 \tag{13.43}$$

$$X_2 = \frac{A^2RX_3}{X_3 + R} + Q = \frac{4 \cdot 2 \cdot 27.843}{27.843 + 2} + 20 = 27.464 \tag{13.44}$$

$$X_1 = \frac{4 \cdot 2 \cdot 27.464}{27.464 + 2} + 20 = 27.457 \tag{13.45}$$

$$X_0 = \frac{4 \cdot 2 \cdot 27.457}{27.447 + 2} + 20 = 27.457 \tag{13.46}$$

Given those values for X_k, the values for K_k and the control signal u_k can then be determined and this has been done for each of the four cases where simulations were also run to yield the overall tables of results below.

Cases 1 and 2 involve an unstable process with system matrix $A = 2$ whose free discrete time response to the initial condition $x_0 = 100$ would be as follows: $x_1 = 200$, $x_2 = 400$, $x_3 = 800$, $x_4 = 1600$.

As the tables and graphs will confirm, that cases 1 and 2 require quite vigorous control activity.

Case 1 $A = 2$, $Q = 20$, $R = 2$

k	X_k	K_k	u_k	x_k	J_k
4	100	–	–	$9.7 \cdot 10^{-3}$	$4.69 \cdot 10^{-3}$
3	27.843	1.9608	−0.4845	0.2471	$8.5 \cdot 10^{-1}$
2	27.464	1.8660	−3.4399	1.8435	$4.67 \cdot 10^{1}$
1	27.457	1.8642	−25.315	13.579	$2.53 \cdot 10^{3}$
0	25.457	1.8642	−186.42	100	$1.37 \cdot 10^{5}$

Case 2 $A = 2$, $Q = 2$, $R = 20$

k	X_k	K_k	u_k	x_k	J_k
4	100	–	–	3.4552	$5.97 \cdot 10^{2}$
3	68.667	1.6667	−17.276	10.366	$3.69 \cdot 10^{3}$
2	63.955	1.5489	−35.589	22.977	$1.69 \cdot 10^{4}$
1	62.942	1.5236	−73.475	48.226	$7.32 \cdot 10^{4}$
0	62.709	1.5177	−151.77	100	$3.14 \cdot 10^{5}$

Cases 3 and 4 both use the model $A = 0.5$. That is an open loop stable process that without control would successively halve the initial condition $x_0 = 100$ to yield the discrete time response $x_1 = 50$, $x_2 = 25$, $x_3 = 12.5$, $x_4 = 6.25$. Thus, the process has quite a lot of inherent disturbance correction ability and it can be expected that significantly less control action will be required than for the unstable case ($A = 2$).

Case 3 $A = 0.5$, $Q = 20$, $R = 2$

k	X_k	K_k	u_k	x_k	J_k
4	100	–	–	$8.65 \cdot 10^{-5}$	$3.74 \cdot 10^{-7}$
3	20.490	0.4902	$-4.323 \cdot 10^{-3}$	$8.818 \cdot 10^{-3}$	$7.97 \cdot 10^{-4}$
2	20.456	0.45554	$-9.034 \cdot 10^{-2}$	0.1983	$4.02 \cdot 10^{-1}$
1	20.455	0.45547	−2.0283	4.4533	$2.03 \cdot 10^{2}$
0	20.455	0.45547	−45.547	100	$1.023 \cdot 10^{5}$

Case 4 $A = 0.5$, $Q = 2$, $R = 20$

k	X_k	K_k	u_k	x_k	J_k
4	100	–	–	0.6057	$1.834.10^{1}$
3	6.1667	0.41667	−3.0284	7.2681	$1.629.10^{2}$
2	3.1783	0.11783	−2.241	19.018	$5.748.1^{2}$
1	2.6856	0.06856	−3.0223	44.081	$2.609.10^{3}$
0	2.5919	0.05919	−5.919	100	$1.296.10^{4}$

Inspection of the tables above shows the effect of the Q, R matrices on the responses with case 1 having energetic control action and corresponding rapid change in the state variable x, where, in contrast, the control action in case 4 is much less and the rate of reduction in the value of x is not much greater than would occur 'naturally' because of the exponentially decaying free response of the open loop system.

Note: Source material and suggestions for further reading to support the topics of this chapter will be found in Chapter 18. See in particular Section 19.8.

Chapter 14

State estimation: observers and the Kalman filter and prediction

14.1 Introduction

Most of this chapter will be devoted to methods for determining the current value of a not directly measurable state vector from measurements of the outputs y and the inputs u of some linear system. If the system is observable, then, by definition, the state can be determined from linearly processed measurements of y. If the state is varying under the influence of active inputs u, that effect may be calculated exactly, given an exact process model. Such an arrangement, with noiseless measurements and with an exact model available, is usually called a state observer.

A more realistic problem situation, in which models are known only approximately and measurements are subject to noise, goes by the name state estimation and is the main topic that follows.

14.2 The separation principle

A key structural result that allows optimal state estimation to be considered as a dual to optimal control is the *separation principle* (Kalman, 1960). The paper shows that the optimal feedback control of a process whose state is not available may be separated, without loss of optimality, into two linked tasks

(i) that of optimally estimating the state x by a state estimator to produce a best estimate $\hat{x}$ and
(ii) that of providing optimal state feedback based on the estimated state $\hat{x}$, rather than the unavailable true state x.

The two sequential problems form a pleasing pair with symmetry in that, assuming the original process has the equation

$$\frac{dx}{dt} = Ax + Bu, \quad y = Cx$$

then the estimator involves multiplication by the matrix $(A - K_{est}C)$, whereas optimal feedback involves multiplication by the matrix $(A - BK_{control})$, where K_{est}

and $K_{control}$ are the optimal estimation and feedback gains, respectively, that need to be chosen to give overall optimal feedback control.

14.3 State estimation: what it attempts to do

Many powerful feedback control strategies require the use of state feedback (Figure 14.1). However, in many important practical cases the complete state vector is not available to be fed back (certain states are said to be inaccessible). In such cases, a state estimator may be used to reconstruct the state vector from a measured output (Figure 14.2).

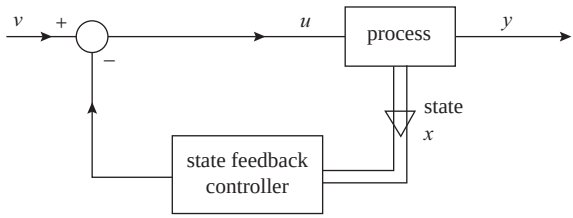


Figure 14.1 Application of state feedback

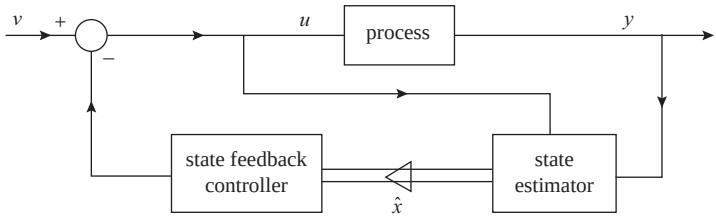


Figure 14.2 Application of state feedback when the state is inaccessible: a state estimator reconstructs an estimate $\hat{x}$ of the true state x

14.4 How a state estimator works: the Kalman filter

We assume that at time $t = 0$, the state x is exactly known, with value x_0. We have a process model that, given x_0, can make a model-based prediction T seconds into the future, to yield the prediction $x_p(T)$.

We also have a measurement y and a known relation $x_m = \alpha y$, applying at all times. In particular, we have $x_m(T) = \alpha y(T)$.

Both the model used for prediction and the measurement y are assumed to be subject to errors. Thus, we have, at time T, two estimates of the true state $x(T)$.

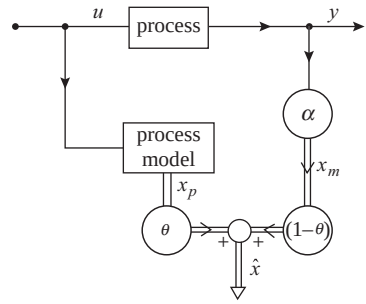


Figure 14.3 Simple illustration of the principle of the Kalman filter

These are

$x_p(T)$, predicted by a model
$x_m(T)$, based on measurement

The best estimate of $x(T)$ is denoted $\hat{x}(T)$ and is determined by the relation

$$\hat{x}(T) = \theta x_p(T) + (1 - \theta)x_m(T)$$

(where θ is a coefficient between 0 and 1 whose value is determined by the relative statistical confidence that can be placed in the accuracy of the model and of the measurement (see Figure 14.3)).

A whole armoury of techniques, under the generic name Kalman filter, deal with all aspects of the application to different situations.

14.5 The Kalman filter: more detail

Figure 14.4 shows the Kalman filter connected to a process with inaccessible state vector $x(j)$. It is assumed that the process state and the measurement vector $y(j)$ are corrupted by white Gaussian noises $w(j)$, $v(j)$, respectively, with diagonal covariance matrices Q, R.

This means that the noise signals satisfy the following:

$$\mathcal{E}[w(j)] = \mathcal{E}[v(j)] = 0$$
$$\mathcal{E}[w(j)w^{\mathrm{T}}(k)] = Q, \text{ for } j = k, \quad \mathcal{E}[w(j)w^{\mathrm{T}}(k)] = 0, \quad \text{for } j \neq k$$
$$\mathcal{E}[v(j)v^{\mathrm{T}}(k)] = R, \text{ for } j = k, \quad \mathcal{E}[v(j)v^{\mathrm{T}}(k)] = 0, \quad \text{for } j \neq k$$

and independence of the noise signals is also implied and assumed, i.e.

$$\mathcal{E}[w(j)v^{\mathrm{T}}(k)] = 0, \quad \text{for all } j, k$$

and finally, the initial condition $x(0)$ is assumed to be a further random variable as follows:

$$\mathcal{E}[x(0)] = x_0$$

$$\mathcal{E}[x(0)\ x^{\mathrm{T}}(0)] = P(0)$$

where $\mathcal{E}$ denotes expected value.

The process is assumed to have the model:

$$x(j) = Ax(j-1) + Bu(j-1) + Ew(j-1)$$

$$y(j) = Cx(j) + v(j) \tag{14.1}$$

At time $t = (j-1)T$, the discrete time, linear model $[A, B, C]$ is supplied with a previous best estimate of the state, designated as $\hat{x}(j-1/j-1)$ and with a measured value of $u(j-1)$. Then, noting that $\hat{x}(j/j-1)$ means a prediction of $x(j)$ made at time $(j-1)$ and using the equation

$$\hat{x}(j/j-1) = A\hat{x}(j-1/j-1) + Bu(j-1)$$

$$\hat{y}(j) = C\hat{x}(j/j-1) \tag{14.2}$$

a one-step ahead prediction of the state and of the corresponding output is made. (Note that since w, v are Gaussian, they have zero mean and hence do not appear in the prediction (14.2)). When time $t = jT$ is reached, the output prediction error $\hat{y}(j)$ can be calculated from the equation

$$\hat{y}(j) = y(j) - \hat{y}(j) \tag{14.3}$$

Finally, we obtain the current best estimate $\hat{x}(j/j)$ by adding to the model prediction $\hat{x}(j/j-1)$, a correction term, proportional to $\hat{y}(j)$, according to the equation

$$\hat{x}(j/j) = \hat{x}(j/j-1) + K(j)\hat{y}(j) \tag{14.4}$$

$K(j)$ is called the Kalman gain matrix and it must be chosen so that the estimates $\hat{x}(j/j)$ are optimal in some sense. However, before considering optimality, it can be seen from the block diagram that the Kalman gain is within a feedback loop, and the wider question arises: Will the sequence $[\hat{x}(j/j)]$ converge to $x(j)$? If so, how quickly will it converge? Will there be a bias in the estimate? How accurate must the process model be? What if the process is non-linear? How accurately must covariance matrices Q, R be specified? What if w, v are non-Gaussian? What time step T needs to be chosen for the discretisation? What if the process is time-varying or some of its parameters are not known a priori?

The practical questions will be considered later, but now we return to the question of choosing the optimal gain matrix $K(j)$.

14.6 Obtaining the optimal gain matrix

From (14.3) and (14.4),

$$\hat{x}(j/j) = \hat{x}(j/j-1) + K(j)[y(j) - C\hat{x}(j/j-1)] \tag{14.5}$$

Then using (14.2),

$$\hat{x}(j/j) = A\hat{x}(j-1/j-1) + Bu(j-1) + K(j)[y(j) - C\hat{x}(j/j-1)] \tag{14.6}$$

The state estimation error is defined as

$$\tilde{x}(j) = x(j) - \hat{x}(j/j) \tag{14.7}$$

but

$$x(j) = Ax(j-1) + Bu(j-1) + w(j-1) \tag{14.8}$$

and

$$y(j) = Cx(j) + v(j) = C[Ax(j-1) + Bu(j-1) + w(j-1)] + v(j) \tag{14.9}$$

Substituting (14.9) into (14.6) yields

$$\tilde{x}(j) = [I - K(j)C][A\tilde{x}(j-1) + Ew(j-1)] - K(j)v(j) \tag{14.10}$$

Define

$$P(j) = \mathcal{E}[\tilde{x}(j)\tilde{x}(j)^{\mathrm{T}}]$$

where $\mathcal{E}$ indicates expected value and where the superscript T indicates transpose. P is a covariance matrix that indicates the accuracy of the state estimation. The system of Figure 14.4 is linear and the disturbing signals are Gaussian. Under these conditions, the solution of (14.10) to yield the gain matrix $K(j)$ that minimises the estimation error is yielded by application of classical optimal control theory. In fact, the optimal estimation problem and the optimal control problem lead to the same equations, and for this reason the two problems are often considered to be duals.

After some manipulation whose detail is omitted (but see for instance Åström (2008) or for a more comprehensive derivation (that does not assume that the filter should be recursive) Fieguth (2011), Section 4.2), the optimal gain matrix is found to be

$$K(j) = M(j)C^{\mathrm{T}}(CM(j)C^{\mathrm{T}} + R)^{-1}$$

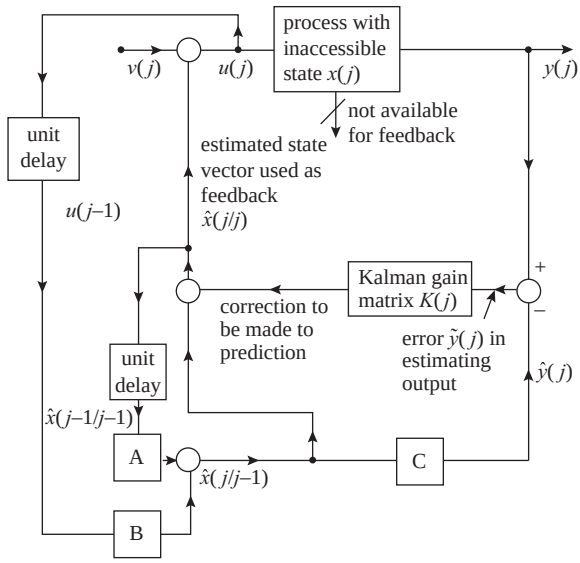


Figure 14.4 The Kalman filter connected to a process with inaccessible state so as to feed back an estimate of the process state for closed loop control

where

$$M(j) = AP(j-1)A^{\mathrm{T}} + EQE^{\mathrm{T}}$$
$$P(j) = (I - K(j)C)M(j)$$

Notice that the equations for $K(j)$ contain no measured data and that therefore they may be solved for all values of j in advance, off-line, if need be.

The optimal state estimator is given by

$$\hat{x}(j/j) = (1 - K(j)C)[A\hat{x}(j-1/j-1) + Bu(j-1)] + K(j)y(j)$$

and we can return to Figure 14.4 to understand how the algorithm is coupled in real time to the process whose state is to be estimated.

14.7 Prerequisites for successful application of the Kalman filter in the form shown in Figure 14.4

(i) There must exist a 'sufficiently accurate' linear, discrete time, time-invariant process model (A, B, C, E).
(ii) The disturbing noises v, w must be Gaussian with zero mean and their covariance matrices R, Q must be known.
(iii) Online computing power must be available, capable of performing the necessary calculations within a time interval that will usually be dictated by the process dynamics.

14.8 Discussion of practical points arising

(i) *Time-varying processes*: The Kalman filter theory is applicable directly to a time-varying process $\{A(j), B(j), C(j), E(j)\}$.

(ii) *Continuous time processes*: Most processes to which the Kalman filter is to be applied will operate in continuous time. Such processes will usually be approximated by discrete time models, or, at least in theory, the Kalman–Bucy filter (1961) could be applied. The discretisation process is easily performed, but care must be taken not to introduce spurious errors during the discretisation process.

(iii) *Non-linear processes*: Most important processes are non-linear and the usual procedure is to use a different linear approximation $\{A(j), B(j), C(j), E(j)\}$ to represent the process at each time step jT. This procedure is equivalent to linearising about a time trajectory. The filter operating in the way described is usually referred to as the extended Kalman filter.

(iv) *Complex processes*: The Kalman filter for a complex process will, of necessity, be based around a low-order approximation to the process.

(v) *Processes that vary with time in unknown ways*: A process that is changing with time may have some or all of its model parameters estimated numerically in real time from measured process data. The procedure may be performed separately from the Kalman filtering operation. Alternatively, the required model parameters may be estimated, along with the process states, using the Kalman filter. In essence, such model parameter estimation is performed by relabelling as state variables those parameters that are to be estimated. Such relabelling clearly introduces artificial non-linearities into the process equations. These non-linearities are dealt with by linearisation in the same way as when the process equations are inherently non-linear.

(vi) *Non-Gaussian disturbance signals*: A non-Gaussian signal (say, $r(z)$) can be treated by synthesising a filter transfer function (say $G(z)$) such that

$$r(z) = G(z)v(z)$$

where $v(z)$ is a white noise signal.

Thus, by adding a new element G to the process model, the requirement that v shall be a Gaussian signal of zero mean may be met. The element G used in this way is sometimes referred to as a colouring filter.

(vii) *Disturbing signals v, w have covariance matrices R, Q that are unknown*: Experimental observation of signals may give some quantitative information on the numerical values for R, Q. Simulation studies of the Kalman filter coupled to a process model will usually give considerable guidance of the choice of R and Q since these matrices affect the convergence of the estimate of the state to its true value (the true value of the state is, of course, known in a simulation study). By performing simulation runs with different choices of R and Q, it is usually possible to choose compromise values that will yield good convergence over a wide range of conditions.

14.8.1 Use of the innovation sequence to modify R and Q

The sequence $\{\tilde{y}(j)\}$ is known as the innovation sequence. Under ideal conditions, when all the initially stated assumptions are satisfied, the innovation sequence will be Gaussian with zero mean. It therefore follows that bias or non-Gaussianness in the innovation sequence may be used in a corrective online feedback loop to modify parameters, e.g. in the colouring filter $G(z)$ described in list (vi) above.

The Kalman filter ***idea*** (combining a model-based estimate with a measurement-based estimate) is remarkably robust in industrial situations where signals are subject to drift, do not have zero mean and whose variances are unknown.

In such cases, the Q and R matrices perform, to some extent, the role of tuning parameters, being adjusted on site according to results obtained.

14A A Matlab demonstration of the Kalman filter at work, estimating a true system output from the noise-corrupted output

This is a summary: The full demonstration is available at the quoted web address*. It shows the calculation steps for both a steady-state and a time-varying Kalman filter applied to the third-order discrete time process:

$$x(k+1) = Ax(k);\ y(k) = Cx(k)$$

$$\text{with } A = \begin{bmatrix} 1.13 & -0.49 & 0.11 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} -0.38 \\ 0.59 \\ 0.52 \end{bmatrix}$$

$$C = [\,1 \quad 0 \quad 0\,]$$

Noise covariances are preset at Q (process noise) $= 2.3$, R (measurement noise) $= 1$

Note (Figure 14.5) that there is a process with a significant process noise ($Q = 2.3$) whose supposed real-world output y' is corrupted by inevitable measurement noise with variance $R = 1$.

This being a simulation, we have access to the 'true' process output y that of course is carrying the effects of the significant process noise.

The task of the Kalman filter, connected only symbolically in the figure, is to remove the measurement noise and to generate a best estimate $\hat{y}$.

Using Matlab command 'Kalman' yields the steady-state Kalman gain

$$K = \begin{bmatrix} 0.53 \\ 0.01 \\ -0.48 \end{bmatrix}$$

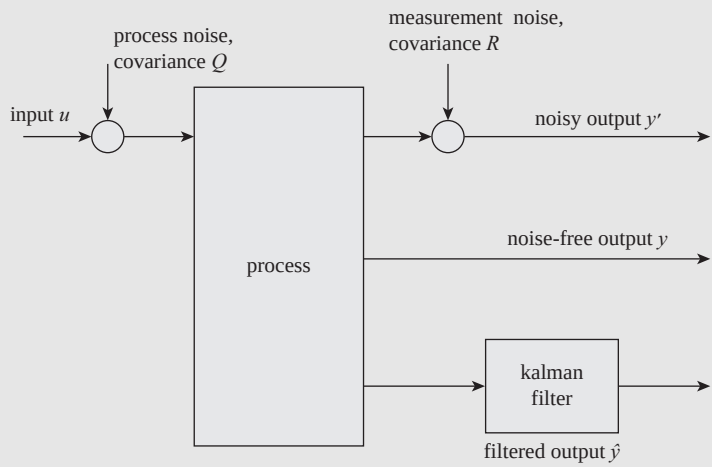


Figure 14.5 Main input-output variables considered in the Matlab Kalman filter demonstration

In the demonstration, simulation plots show the time behaviour of the three output variables in response to a sinusoidal input. From the simulation results:

(i) The covariance of the measurement noise $y-y'$ was found to be 1.1138 (compare with $R = 1$).
(ii) The covariance of the estimation error $\hat{y} - y'$ when using the steady-state Kalman filter was found to be 0.4309. Compare with (i).
(iii) The time-varying Kalman filter converged to the same gain K as the steady-state version after about five time steps.
(iv) Nevertheless, the performance of the time-varying filter did show a small improvement over the steady-state version, yielding an estimation error covariance for $y-y'$ of 0.4303.

*The demonstration can be found on the Mathworks website by seeking Control System Toolbox – Kalman Filter Design Demo.

14.9 Prediction and predictive control

Prediction is discussed briefly here because accurate estimation of process states from noisy data usually requires a model-based prediction, say $x(j/j-1)$ meaning a one-step ahead prediction that, in a Kalman filter approach, will be combined statistically with an instantaneous measurement-based estimate to produce an optimal overall estimate $x^*(j)$.

Since almost every human action is based on prediction, even if it is unconscious prediction, it is no surprise that many controllers have been designed (even the D in PID controllers is doing it!) using predictive models. One such example is discussed in interlude 14B.

14B Model Predictive Control (MPC), also know as predictive-iterative control

In this approach, pioneered by Coales and Noton (1956), a fast online model, preferably one hundred or more times faster than the process, is fed with trial control inputs and in response generates a large number of possible future trajectories. Rapid automatic evaluation of the trajectories leads to a choice of a best strategy to actually implement.

It is good practice, and is usual in most forecasting, to use a prediction horizon that is considerably longer than the implementation horizon.

The strategy can produce performances close to time-optimal for the control of switched systems. Figure 14.6 illustrates the main ideas.

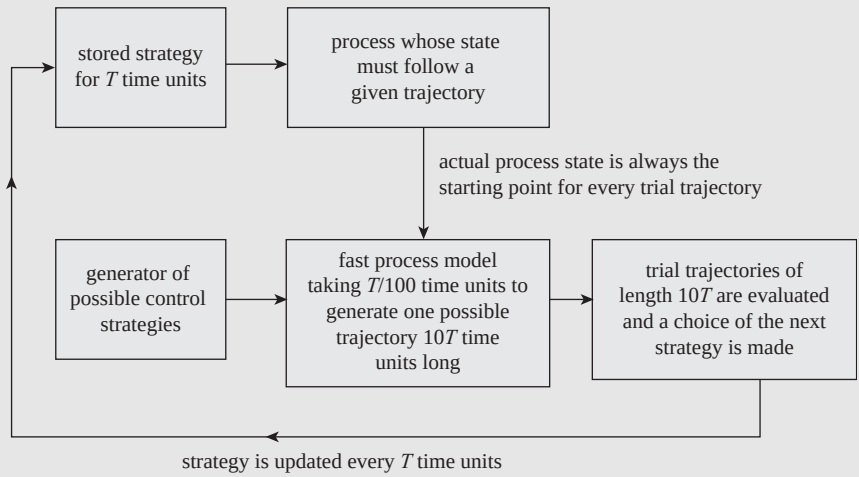


Figure 14.6 The principle of Model Predictive Control (MPC). A fast model receives possible future strategies and rapidly predicts how successfully each would perform. This is performed in real time with every trial trajectory having as its initial condition the real current location. Like many empirical strategies, this can be a very successful approach to real problems

Note: Source material and suggestions for further reading to support the topics of this chapter will be found in Chapter 18. See in particular Section 18.6.

Chapter 15

An introduction to robust control design using H_∞ and related methods

15.1 Motivation and introduction

Many promising optimisation techniques have, in the past, failed to live up to their promise; one of the most important reasons for this failure being the lack of robustness in the methods. In particular, very complex plant models were often produced and then naively assumed to be accurate representations of the real world. The inevitable mismatches between the assumed (let us say, nominal) models and the real-world processes destroyed the viability of many approaches.

H_∞ approaches, by specifically taking into account modelling uncertainty, and doing so in a worst-case sense, allow complex control design problems to be solved in a theoretically rigorous way while guaranteeing robustness of the implemented solutions over a pre-specified range of model incorrectness or (equivalently) of process variability.

Here, we review the linear spaces that underlie much of the modern operator-based control theory with particular emphasis on the theory underlying H_∞ approaches. Some of the H_∞ control design methodology is then introduced in very simple terms to establish the basic principles.

Controller design by *interactive loop shaping* of sensitivity functions in the frequency domain is demonstrated through a tutorial worked example. This is a powerful, yet intuitively transparent design approach.

The *ν gap metric* is explained. It is a useful visualisable quantitative tool for grouping and classifying system models in terms of their expected behaviour under closed loop optimal control. Perhaps the biggest gap between theory and practice in optimal control has been caused by the near impossibility of providing an industrially meaningful cost function that was also mathematically tractable. With the arrival of *Linear Matrix Inequalities* (*LMIs*) software routines, it is now possible to insert a realistic cost function into an optimisation algorithm and, using standard Matlab or similar routines, to deliver the required controller designs using efficient numerical search methods based on *interior point methods*.

So-called *μ-synthesis methods* are introduced and explained. In essence, they are basic H_∞ controller design algorithms with additional subdivision of the model/controller block diagram in such a way that the design parameter μ emerges as a

key quantity and, most importantly, the designer using the algorithm is able to input directly, in engineering units, the magnitudes of the uncertainties that the sought-for robust controller must be able to tolerate.

15.2 Hardy spaces (H_p, $p > 0$ spaces) and their relevance to control problems

Hardy spaces (see interlude 15A) are of value in control problem formulation since they provide a rigorous theoretical foundation for representing the Laplace or Fourier transform models of linear dynamical systems together with an easy link to equivalent time-domain representations. The spaces H_2 and H_∞ are the spaces of primary interest.

Linear multivariable optimisation problems with quadratic cost functions can be formulated and solved very satisfactorily in an H_2 setting in a coherent way. Optimisation in an H_2 setting can in fact be considered as a more modern replacement for linear quadratic Gaussian (LQG) approaches. Note that, by convention, the H_2 norm is applied to transfer functions/transfer matrices and the L^2 norm to time functions. H_∞ is the Hardy space of all stable linear time-invariant continuous time system models, and the H_∞ norm is a scalar measure of the upper limit of the gain of a transfer function $G(\omega)$ of a matrix of such transfer functions as frequency ω is varied.

15.2.1 The suffix p

The suffix p indicates that the space Hp is furnished with the p norm, so that given any element x (and such elements will normally be functions) belonging to Hp, we can measure the 'size' of x by a norm such as

$$\| x \|_p = \left(\int_0^\infty |x(t)|^p \right)^{1/p} \tag{15.1}$$

15.2.2 Elementary illustration: the effect of choice of p on the nature of the norm

Figure 15.1 shows a time function. We evaluate its norm using (15.1) for values of $p = 1, 2, \ldots, 256$ and have plotted the results in Figure 15.2. We observe (as emphasised by the starred arrow in Figure 15.2) that as $p \to \infty$, $\|f\|_\infty \to f_{max}$; in other words, the H_∞ norm of a function simply measures the peak value of the function over a specified interval. H_∞ then is a convenient function space where the functions are normed according to their maximal values (strictly suprema).

In our example of Figure 15.1 and the plot of Figure 15.2, it was the case that

$$\| f \|_p \geq \| f \|_q, \quad p > q$$

and this is a general rule with equality holding only for functions of constant magnitude.

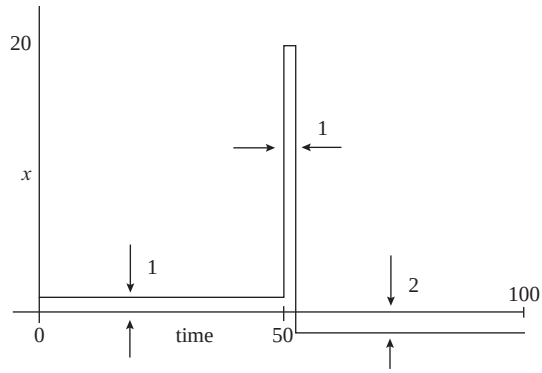


Figure 15.1 Test function to illustrate the effect of the choice of p

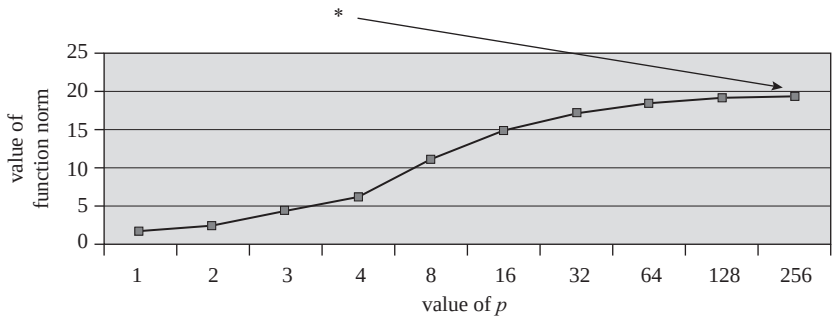


Figure 15.2 Illustrating how the norm of the function of Figure 15.1 is affected by the choice of p

15.2.3 Non-elementary aside

Note carefully though that two functions that differ only at isolated points (i.e. they differ only on a set of measure zero) will have identical norms. This point is of considerable mathematical interest in the theory of Lebesgue integration.

In control applications, it will be rare to use values of p other than $p = 1$, 2 or ∞.

The choice of $p = 1$ leads to 'integral of absolute error' criteria that are sometimes used in loop tuning criteria. The choice of $p = 2$ leads to quadratic criteria that are ubiquitous since they lead to convexity and tractability, convexity being perhaps second only to linearity as a desirable quality. Note (Figure 15.3) how the unit ball satisfying

$$\|x\|_p = 1$$

looks for various values of p. From Figure 15.3 it can be seen that the unit ball has the highly desirable property of strict convexity only for the case $p = 2$.

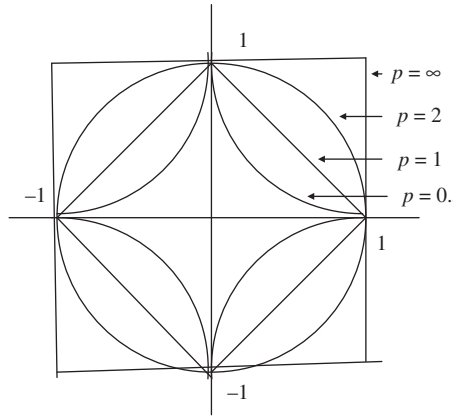


Figure 15.3 The shape of the unit ball in real two-space for different values of p

What about H_p for $p < 1$? It will be found that when $p < 1$, H_p is no longer a normed space since the hoped-for norm fails the triangle inequality (which is one of the necessary conditions that a norm must satisfy):

$$\| x_1 \| + \| x_2 \| \geq \| x_1 + x_2 \|$$

as the following simple example for the real plane with $p = 0.5$ demonstrates.

Let

$x_1 = (1, 0)$, $x_2 = (0, 1)$ so that $x_1 + x_2 = (1, 1)$

Then,

$$\| x_1 \| = \| x_2 \| = 1 \quad \text{but} \quad \| x_1 + x_2 \| = \left(\sum_{i=1}^{2} |x_i|^{1/2} \right)^2 = 4$$

which contravenes the triangle inequality.

15.3 A simple view of H_∞ control loop design

15.3.1 Guaranteed stability of a feedback loop

Zames (1976, 1981) is credited with founding H_∞ theory around the basic idea that a control loop can be represented by operators whose maximum gain across all frequencies (speaking loosely) can be represented by the H_∞ norm.

It is a key result of elementary control theory that the loop of Figure 15.4 will be input–output stable provided either

(1) the gain of the *GD* combination is less than unity at all frequencies or
(2) the phase lag of the *GD* combination is less than 180° at all frequencies.

But we can now express condition (1) in H_∞ language as the closed loop of Figure 15.4 can be guaranteed input–output stable provided that $\| G(\omega)D(\omega) \|_\infty < 1$

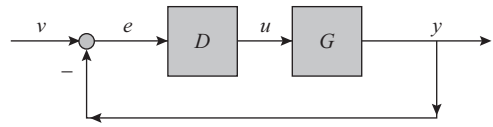


Figure 15.4 Basic feedback control loop

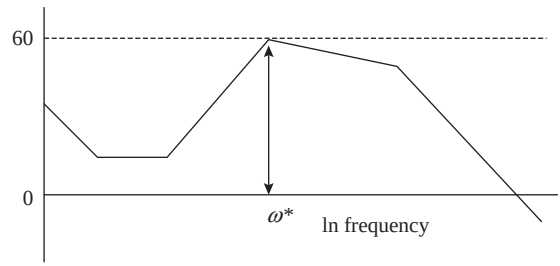


Figure 15.5 A possible Bode magnitude plot for $D(s)G(s)$

(it being understood that for this example the H_∞ norm represents the maximum gain at any frequency).

Figure 15.5 shows a Bode magnitude sketch for a possible $G(s)D(s)$ combination.

It has a peak value of around 60 dB at frequency ω^*. This means that

$$\frac{\| y(s) \|}{\| e(s) \|} \leq \| G(s)D(s) \|_\infty \approx 4.1$$

(converting 60 dB to a linear gain) and it can be seen that the H_∞ norm is simply the peak value of the Bode magnitude plot.

15.3.2 Robust stability of a closed loop

Consider next the closed loop of Figure 15.6 in which G represents the best available (nominal) process model and ΔG represents a deterministic model of the maximum model uncertainty. This closed loop can be guaranteed stable provided that

$$\| (G(\omega) + \Delta G(\omega))D(\omega) \|_\infty < 1$$

and this inequality is the very essence of robust control design using H_∞ methods.

Quoting Lunze (1989), it can be seen that $D(s)$ might be considered to be a stabilising common controller for the family of process models that exist within the $G + \Delta G$ envelope.

What more needs to be done or discussed before H_∞ ideas can be applied in anger? Very roughly the following:

(i) Above, we considered only input–output stability – below we shall consider total internal stability. This will involve considering a matrix of four transfer functions even in the single-input, single-output case.

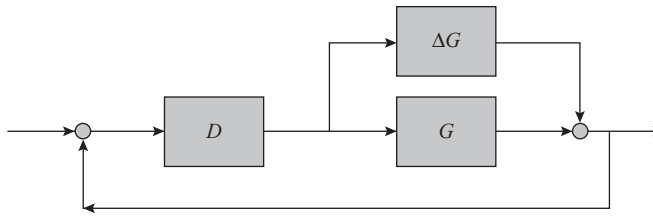


Figure 15.6 Feedback control of a process with uncertainty ΔG

The implication of (i) above is that we need a method for defining the H_∞ norm of a matrix, not necessarily square, of transfer functions. Of course, matrix transfer functions are also involved in the generalisation to multivariable problems.

(ii) Ensuring stability by simply keeping loop gain below some peak value is only an important elemental idea. A complete design procedure will ensure good dynamic and steady state responses and rejection of disturbances despite process model uncertainties while guaranteeing stability. Such design procedures will need to trade stability margins with performance targets, using high gains in those regions of the frequency spectrum where performance is critical with carefully chosen lower gains where stability is most critical.

(iii) As would be expected, making a deterministic model of uncertainty is bound to be difficult since uncertainty is sure to be poorly defined and difficult to pin down. Three structures are explained below to allow the modelling of different types of uncertainty.

(iv) We need to be able to define numerical algorithms for calculation of the H_∞ norms of process model/controller combinations.

15.4 Total internal stability and design for disturbance rejection

15.4.1 Setting the scene

Consider a dynamic process with impulse response $g(t)$. The output of such a process in response to an input $u(t)$ is given by the usual convolution integral

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau \tag{15.2}$$

and provided that the convolution integral is bounded on $L^2[0,\infty)$, then we can take Laplace transforms and write

$$y(s) = G(s)u(s)$$

where the transfer function $G(s)$, being bounded, belongs to H_∞ and

$$\| G \|_\infty = \sup_{\|u\|_2<1} \| y \|_2$$

Now consider the feedback loop of Figure 15.7 where a process of transfer function G is in a loop with a controller of transfer function D.

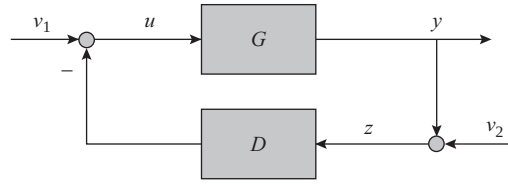


Figure 15.7 Closed loop control; G is a process and D a controller

It is easy to show from the diagram that the following matrix relation holds

$$\begin{bmatrix} u \\ z \end{bmatrix} = \begin{bmatrix} \dfrac{1}{1+GD} & \dfrac{G}{1+GD} \\ \dfrac{D}{1+GD} & \dfrac{1}{1+GD} \end{bmatrix} \begin{bmatrix} v_2 \\ v_1 \end{bmatrix} \tag{15.3}$$

and the feedback system is internally stable if all four of the transfer functions within the matrix of (15.3) belong to the space H_∞. A sufficient condition for this is that

$$|GD|_\infty < 1 \tag{15.4}$$

15.4.2 Robust stability

Although a system such as the one in Figure 15.7 is guaranteed to be stable under the condition (15.4), there is for all practical systems a further requirement that the system should remain stable despite variations from nominal in the process G.

A feedback control system that can be guaranteed to remain stable under a specified range of process perturbations is said to possess robust stability. What we are discussing is the very common situation where the real process and its model differ by some margin, either because the process varies in quite complex ways, whereas the model is constant, or because the model is a considerable simplification of the real-world process.

Some examples: the characteristics of a strip rolling mill differ markedly according to the width, thickness and metallurgy of the product being rolled; the stabilisers of a ship interact with the effect of the rudder and vary according to ship speed; an industrial biological process varies in a complex way as the batch progresses. For all these examples, no single model can exactly allow for those variabilities. Even with a fixed known process, the modeller will almost always have to neglect effects such as high-order dynamics in the interests of keeping model complexity within bounds.

As the examples hopefully demonstrate, process models can only represent the real process to within some margin of error that we will name ΔG.

We assume the feedback loop for which a robust controller D is to be designed is as shown in Figure 15.7 and we also assume that the perturbation ΔG is bounded

by the H_∞ norm, i.e.

$$\underset{\text{for almost all } \omega}{|\Delta G(\omega)|} \leq |R(\omega)| \tag{15.5}$$

for some R. A key result is that the system will remain internally stable under all perturbations possible within inequality (15.5) if and only if

$$\| RD(1 + GD)^{-1} \|_\infty < 1 \tag{15.6}$$

To allow the concept to be appreciated in a Nyquist diagram context, we rearrange inequality (15.6) into the form

$$R(\omega)|G(\omega)D(\omega)| < |1 + G(\omega)D(\omega)| \tag{15.7}$$

The diagram Figure 15.8 shows how the circle of model uncertainty must not enclose the $-1 + j0$ point, if stability is to be guaranteed in the closed loop system.

The robust control design procedure is then to choose the controller D to satisfy the inequality (15.6) while simultaneously meeting all performance specifications such as response rates, accuracies and disturbance rejection requirements.

It should be noted that

(i) the designer is given no guidance for the choice of controller D except that the frequency-dependent inequality must be observed;
(ii) the choice of a high value for R in an attempt to obtain a high degree of robustness will force down the inequality 'ceiling', resulting in a possibly unacceptable performance. Thus, not surprisingly, the overall design must balance performance and stability requirements.

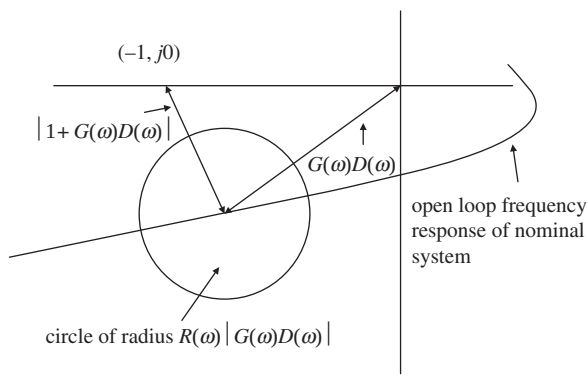


Figure 15.8 Nyquist diagram illustrating the stability inequality (15.7)

15.4.3 Disturbance rejection

Disturbance rejection requirements can be injected into the H_∞ design procedure as follows. Suppose r is a disturbance signal whose effect on system output y is to be minimised and suppose also that $r(s)$ is generated by the transfer function W from any disturbance signal v_1 satisfying

$$r(s) = W(s)v_1(s) \qquad \| v_1(\omega) \|_2 \leq 1 \tag{15.8}$$

Then it can easily be shown that the disturbance effect can be minimised by minimising the quantity

$$\| W(1 + GD)^{-1} \|_\infty \tag{15.9}$$

If we look at the two relations (15.6) and (15.9) relating to robust stability and noise rejection, respectively, another design compromise can be appreciated.

Setting $G = 1$ and $D = k$, i.e. an ultra-simplistic situation to emphasise this point, we have from (15.6) that

$$\frac{k}{k+1}$$

needs to be as small as possible, while from (15.9) that

$$\frac{1}{k+1}$$

also needs to be as small as possible.

The first inequality is asking for k to be as small as possible, whereas the second expression requires k as large as possible. The usual approach to this compromise will be to minimise

$$T = \| D(1 + GD)^{-1} \|_\infty \tag{15.10}$$

over that part of the frequency spectrum where accurate control is most critical and to minimise

$$S = \| (1 + GD)^{-1} \|_\infty \tag{15.11}$$

over that part of the frequency spectrum where disturbance rejection is most important. (S and T are often referred to as the sensitivity coefficient and complementary sensitivity coefficient, respectively.)

This leads to the concept of 'loop shaping' in which the design of the controller D can be viewed as an interactive operation to achieve the best possible performance by satisfying a number of competing frequency-dependent targets and constraints.

Ideally, since we would like

$$T(\omega) = 1 \quad S(\omega) = 0 \quad \text{for all } \omega$$

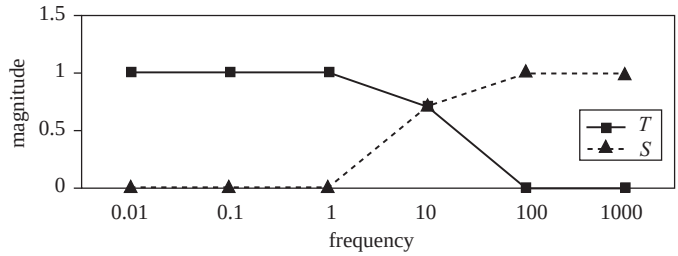


Figure 15.9 The loop shaping concept showing variation of S and T with frequency

this would give perfect following and perfect noise rejection. However, it can be seen that in every case the following limits obtain

$$\lim_{\omega\to\infty}|T(\omega)| = 0$$

$$\lim_{\omega\to\infty}|S(\omega)| = 1$$

so that the design procedure consists in getting the best overall system behaviour within the above constraints. This leads to a controller synthesis methodology sometimes referred to as the mixed sensitivity approach, which results typically in magnitude versus frequency plots for S and T as shown in Figure 15.9.

Note that near-optimal designs will have sharp roll-off characteristics requiring high-order controllers and an iterative interactive design approach such as used by Kwakernaak (1993) where a detailed example is worked through.

Because of the simple correspondences between time and frequency domain properties of H_∞ spaces, design approaches can be expressed and utilised equally well in the time domain using state space approaches.

15.5 Robust control design using a mixed sensitivity H_∞ loop shaping approach: worked example

(I am pleased to acknowledge that this example has been supplied by Dr Victor Becerra of Reading University.)

This generic worked example will show how a robust controller can be designed to achieve a required level of performance for a single-input, single-output process. The generalisation to the multivariable case is usually straightforward. It will be seen that the plots on which the design will be based are magnitude plots with no mention of phase plots. (That is always possible for minimum-phase systems whose phase plots are always completely determined by their magnitude plots.) However, phase angle considerations do constrain any loop-shaping that calls for the system sensitivity curves to climb or fall very steeply. A further disappointing and limiting consequence of Bode's 'laws' is what has been sometimes called

the 'Robust/Fragility trade-off'. This consequence means that extra robustness designed into a system over one range of frequencies will inevitably have to be repaid by accepting an increase in fragility over some other range of frequencies. (Yet another NFL (No Free Lunch) theorem to add to the list!)

There are several unavoidable constraints and compromises when a high-performance robust control loop is to be designed in the presence of uncertainty. The need for high performance implies a high gain controller, whereas the need for a sufficient stability margin implies a cautious design with a low gain. In terms of sensitivity functions, we would like S to be low at all frequencies and T to be equal to unity everywhere. However, the constraining conditions $S + T = 1; S \to 1, T \to 0$ as $\omega \to \infty$ mean that we need to keep S low from zero frequency until the upper limit of the chosen bandwidth is reached and then allow it to increase towards unity. This will allow the (stabilising) curve for T to be moved lower for frequencies above the chosen system bandwidth (Figures 15.10–15.12). As system designers, we hope that the given problem conditions will always allow the performance/robustness specifications to be split on low-frequency/high-frequency lines as just proposed.

Given a process described by the transfer function:

$$G(s) = \frac{s^2 + 0.18s + 0.08}{s^2 + 0.3s + 0.02}$$

We shall design a controller $K(s)$ such that the closed loop system defined by the relation

$$\frac{G(s)K(s)}{1 + G(s)K(s)} = \frac{L(s)}{1 + L(s)}$$

and with the associated sensitivity functions

$$S = \frac{1}{1 + L}, \quad T = \frac{L}{1 + L} \quad [\text{so that } (S + T) = 1]$$

satisfies the following: bandwidth, $w_b = 0.1$ rad/s; steady state error = 1%; sensitivity function S to be as low as possible at low frequencies and to satisfy $|S(jw)| < 1$ for $w > 1$.

Structurally, this can be seen to be an H_∞ problem, since in shaping the loop we shall be asking that $\| w_p S \|_\infty < 1$.

The complementary sensitivity function $T(s)$ is to be bounded above, in order to meet robustness targets, by the given shaping function $w_T = (1.9s + 1)/(10s + 0.33)$.

Since this is an exercise in closed loop shaping, we begin by defining a weighting performance function

$$W_p(s) = \frac{1/M * s + w_b}{s + w_b * A}$$

whose inverse $1/W_p$ will be the required upper bound and shaping function for S.

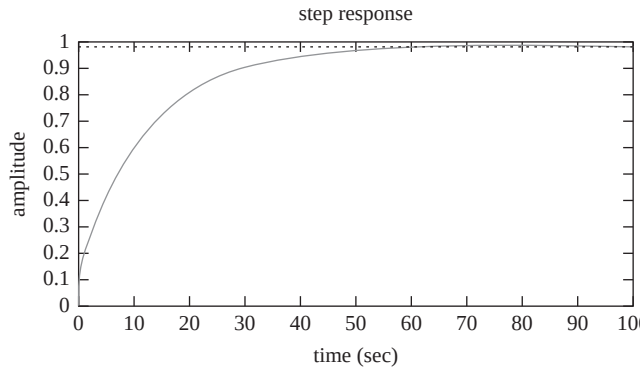


Figure 15.10 The system step response

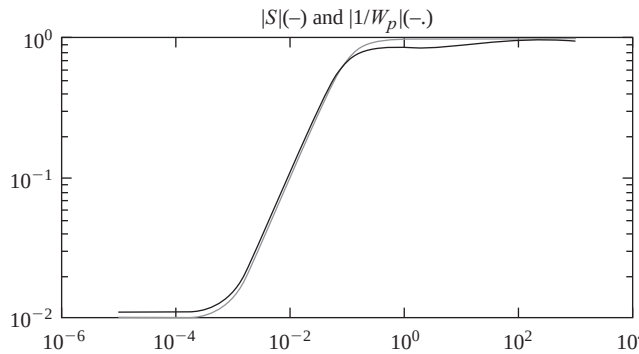


Figure 15.11 The S curve beneath its constraining curve

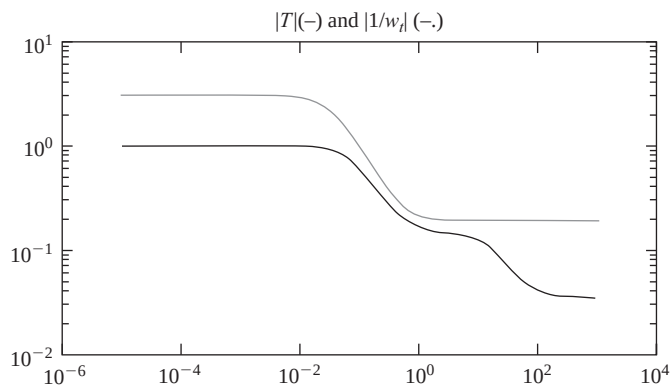


Figure 15.12 The T curve beneath its constraining curve

Setting $w_b = 0.1$ for the bandwidth, $M = 1$ for the high-frequency asymptote and $A = 0.01$ as a suitable low-frequency asymptote, and inserting all required values into the problem produces the following Matlab code:

```
% Plant model
s=tf('s'); G = (s^2 + 0.18*s + 0.08)/(s^2 +0.3*s + 0.02);
% weight Wp
M=1; wb=0.1; A=0.01; Wp = (1/M*s+wb)/(s+wb*A);
% Uncertainty weight Wt.
Wt = (10*s+0.33)/(1.9*s+1)
%Control weight
Wu=0;
% Controller design
[Kss,N,N_norm,INFO]=mixsyn(G,Wp,Wu,Wt); K = tf(Kss);

L=G*K;  % loop transfer function
S=inv(1+L); % Sensitivity
T=1-S;      % complementary sensitivity
figure; step(T,100); figure;
w=logspace(-5,3);

frsp_S =freqresp(S,w);
frsp_T = freqresp(T,w);
frsp_bound_S=freqresp(1/Wp,w);
frsp_bound_T = freqresp(1/Wt,w);
loglog(w,abs(frsp_S(:)),w,abs(frsp_bound_S(:)),'.-' );
title('|S|(-) and |1/Wp| (-.)');figure;
loglog(w,abs(frsp_T(:)),w,abs(frsp_bound_T(:)),'.-' );
title('|T|(-) and |1/Wt| (-.)');
```

and the resulting controller K is

$$K(s) = \frac{0.0361s^4 + 2.344s^3 + 1.918s^2 + 0.412s + 0.02436}{s^4 + 13.78s^3 + 2.541s^2 + 1.09 + 0.001088}$$

Notice that this design approach delivered a fourth-order controller to control the given second-order process. That is fairly typical for this approach. It does suggest that the controller would need to be simulated and evaluated over a range of typical implementation conditions before being taken forward to a real application.

15.6 Specification of the ΔG envelope

Clearly, it will be difficult to specify ΔG in a standard generic form that fits a wide range of applications while remaining mathematically tractable. Considerable effort has been expended on the topic of 'identification for robust control' (Chen, 2000) since the overall credibility of H_∞ design approaches depends on realistic specification of the ΔG envelope.

In the general case, where G will be a matrix transfer function, it will be appreciated that the envelope of allowable uncertainty that we are calling ΔG must be able to represent the effects of, e.g. individual parameters varying significantly, stochastic variation across a range of parameters and neglected dynamics that are in the real process but not in the model. Having noted the difficulty of specifying the ΔG envelope in a case-independent way, we show in Figures 15.13–15.15 the three most common configurations for representing ΔG. The so-called coprime factorisation model (Figure 15.15) allows for the most general modelling of mismatch including the mismatch of neglected dynamics.

15.7 Deriving H_∞ norms from dynamic process models

15.7.1 Singular values and eigenvalues

Singular values and eigenvalues play a central role. Let A be any $m \times n$ matrix. Then singular value decomposition consists in finding orthonormal matrices U, V, i.e. satisfying

$$UU^{\mathrm{T}} = VV^{\mathrm{T}} = I$$

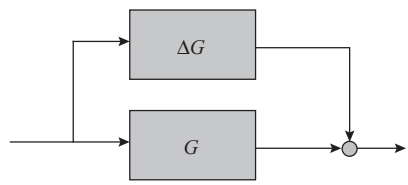


Figure 15.13 Additive uncertainty model $(G + \Delta G)$

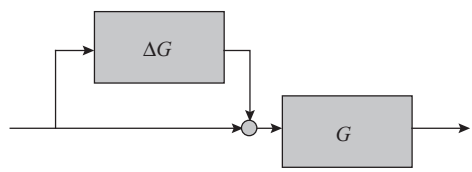


Figure 15.14 Multiplicative uncertainty model $G(1 + \Delta G)$

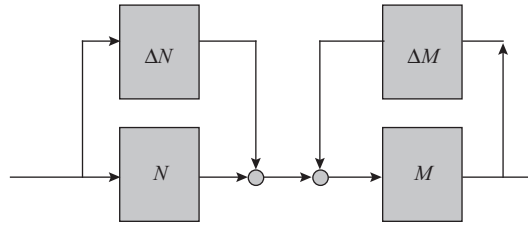


Figure 15.15 Coprime factorisation uncertainty model. Factorising G to form $G = M^{-1}N$ and then perturbing M, N, separately, leads to the diagram shown

and

$$A = USV^{\mathrm{T}}$$

where

$$S = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \tag{15.12}$$

where Σ is a diagonal matrix of non-zero singular values σ_i of A, usually arranged in descending order such that

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$$

Note that the range space $R(A)$ of A is generated by the set

$$\{u_i\}, \qquad i \in [1, r]$$

and the null space $N(A)$ of A by the set

$$\{u_i\}, \qquad i \in [r+1, n]$$

By convention, the largest singular value is denoted

$$\overline{\sigma}$$

and the smallest singular value by

$$\underline{\sigma}$$

Consider the equation

$$y = Ax$$

Then it can be seen that

$$\sigma(A) \leq \frac{|y|}{|x|} \leq \overline{\sigma}(A)$$

and therefore the operator norm is

$$||A|| = \overline{\sigma}(A) \tag{15.13}$$

since the operator norm is defined as

$$||A|| = \sup_{x \neq 0} \frac{|Ax|_2}{|x|_2} \tag{15.14}$$

and if $G(s)$ is a matrix transfer function such as

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \dots & G_{1r}(s) \\ \dots & & & \dots \\ \dots & & & \dots \\ G_{m1}(s) & \dots & \dots & G_{mr}(s) \end{bmatrix} \tag{15.15}$$

then an important result is that

$$||G(j\omega)||_\infty = \sup_\omega \overline{\sigma}(G) \tag{15.16}$$

15.7.2 Eigenvalues of a rectangular matrix A

Consider the equation

$$y = Ax$$

$$||x||_2 = \left(\sum |x_i|^2 \right)^{1/2} = \sqrt{x^* x}$$

where * indicates adjoint. (The adjoint of a vector or matrix is obtained by first transposing and then complex conjugating the elements.) Then

$$|y|^2 = |Ax|^2 = x^* A^* A x$$

Note that a complex-valued matrix $\boldsymbol{A}$ is self-adjoint (Hermitian) if $\boldsymbol{A}^* = \boldsymbol{A}$. Self-adjoint matrices are always diagonalisable and always have real eigenvalues. Note also that $(AB)^* = B^* A^*$.

The matrix $\boldsymbol{A}^* \boldsymbol{A}$ is always square and self-adjoint (Hermitian) since

$$(A^* A)^* = A^* A^{**} A^* A$$

It therefore has real non-negative eigenvalues λ_i. Let these be ordered such that

$$\lambda_1 \geq \lambda_2 \geq \dots \geq 0.$$

15.7.3 Singular values and their relation to eigenvalues

The singular values of a matrix A are defined alternatively as

$$\sigma_i = \sqrt{\lambda_i} \tag{15.17}$$

where λ_i are the eigenvalues of A^*A and

$$\overline{\sigma}(A) = \overline{\lambda}^{1/2}(A^*A) \tag{15.18}$$

where λ is the largest eigenvalue of A.

15.7.4 Relations between frequency and time domains

Of course the domains are linked through the convolution integral

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau \tag{15.19}$$

where $g(t)$ is an impulse response and provided that the convolution integral is bounded on $L^2[0,\infty)$, then we can take Laplace transforms and write

$$y(s) = G(s)u(s) \tag{15.20}$$

where the transfer function $G(s)$ being bounded belongs to H_∞ and

$$\begin{aligned} ||G||_\infty &= \sup_{||u||_2<1} ||y||_2 \\ ||G||_\infty &= \sup_{\omega\in[0,2\pi]} (G(\omega)) \end{aligned} \tag{15.21}$$

and in the time domain

$$||G||_\infty = \sup_{u\neq 0} \frac{||y(t)||_2}{||u(t)||_2} \tag{15.22}$$

and the H_∞ norm on G can be seen to be the usual norm on a mapping from the space of time functions U to the space of time functions Y.

(Note that $||y(t)||_2 = \sqrt{\int_0^\infty y(t)^T y(t)dt)}$

Finally, we note from Parseval's theorem that

$$||\hat{x}||_2 = ||x||_2 \tag{15.23}$$

where x is a time function in $L^2[-\infty, \infty]$ and

$$\hat{x} \quad \text{in} \quad L^2[-j\omega, j\omega]$$

is the Fourier or Laplace transform of x.

15.8 A promising visualisable design tool that works within the H_∞ frame: the ν gap metric

15.8.1 Introduction

The following interesting quotation is from Vinnicombe (2001), as are all the results and examples in this section:

> One of the key aims of using feedback is to minimise the effects of lack of knowledge about the system which is to be controlled. Yet, one clearly needs to know something about that system in order to be able to design an effective feedback compensator for it. So, how accurate need a model be, and in what sense should it be accurate? Or, in other words, 'how much do we need to know about a system in order to design a feedback compensator that leaves the closed loop behaviour insensitive to that which we don't know?'

Let G_1 be the transfer function of a process that is to be controlled and let G_2, G_3 be perturbed versions of G_1. G_1, G_2 and G_3 may be regarded as three possible models of the same process for which a single (robust) controller is sought. The 'distance' between any two processes G_i, G_j in terms of similarity of behaviour when connected into a closed loop can be quantified by the ν gap metric that has the property

$$\delta_\nu(G_1, G_2) = \delta_\nu(G_2, G_1) \in [0, 1] \tag{15.24}$$

An algorithm for the calculation of $\delta\nu$ will be given after an illustrative example.

15.8.2 Simple illustration of the use of the ν gap metric

The following very simple example shows the value of the ν gap metric as a guidepost in deciding how to group processes that may have widely differing open loop responses into clusters that can be successfully controlled by the same controller D. The value of such insight can hardly be overstated.

Define three process models

$$G_1 = \frac{100}{2s+1}$$

$$G_2 = \frac{100}{2s-1}$$

$$G_3 = \frac{100}{(2s+1)^2}$$

Then $\delta\nu(G_1,\ G_2) = 0.02$, whereas $\delta\nu(G_1,\ G_3) = 0.899$, showing that the two models G_1, G_3 are very different from the point of view of the ν gap metric, which (recall) has a maximum value of unity.

One of the conclusions of this worked example is that the two processes G_1, G_2, one stable and the other unstable, would be expected to have very similar closed loop behaviours when controlled by the same controller D.

Fixing the controller $D = -1$ for both cases, we calculate the closed loop transfer functions for the two cases to be

$$\frac{G_1}{1 - G_1 D} = \frac{100}{2s + 1 + 100} = \frac{100}{2s + 101}$$
$$\frac{G_2}{1 - G_2 D} = \frac{100}{2s + 1 + 100} = \frac{100}{2s + 101}$$

confirming the utility of the gap metric in clustering open loop models according to their predicted closed loop behaviour.

15.8.3 More about the two metrics $\delta\nu$ and $b_{G,D}$

Provided that certain continuity conditions relating to right-half-plane poles are satisfied (see Vinnicombe (2001) for details), the following algorithm allows $\delta\nu$ to be calculated:

$$\sqrt{1 - \delta_\nu(G_1, G_2)^2} = \begin{pmatrix} \dfrac{G_2 G_1^*}{1 + G_1^* G_2} & \dfrac{G_2}{1 + G_1^* G_2} \\ \dfrac{G_1^*}{1 + G_1^* G_2} & \dfrac{1}{1 + G_1^* G_2} \end{pmatrix} \tag{15.25}$$

The ν gap metric approach also makes extensive use of another metric, the quantity b, defined by the relation

$$b_{G,D} = \begin{cases} \left\| \begin{bmatrix} G \\ I \end{bmatrix} (I - GD)^{-1} [-D \quad I] \right\|_\infty^{-1} & \text{if } [G, D] \text{ is stable} \\ 0, & \text{otherwise} \end{cases} \tag{15.26}$$

From Figure 15.16 it can be seen that

$$\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} \dfrac{G}{1 - GD} & \dfrac{-GD}{1 - GD} \\ \dfrac{1}{1 - GD} & \dfrac{-D}{1 - GD} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{15.27}$$

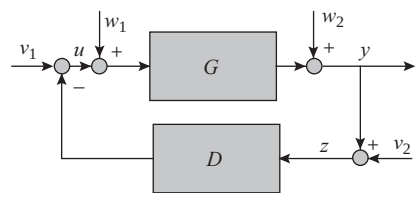


Figure 15.16 Configuration for discussion of the measure $b_{G,D}$

(where for simplicity the weights w_i have been set to zero) and the expression inside the norm sign of (15.26) is the transfer function between

$$\begin{bmatrix} v_1 \\ \hline v_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} u \\ \hline y \end{bmatrix}$$

Properties of $b_{G,D}$

- $b_{G,D} \in [0, 1]$ for any G, D.
- $b_{G,D}$ is a bound for all eight transfer functions linking inputs and outputs in the closed loop.
- $b_{G,D} = b_{D,G}$.
- Let ρ be the 'distance' between the two frequency responses $G(\omega)$, $D(\omega)$. Then $b_{G,D} = \inf\omega\ \rho(G(\omega), D(\omega))$, i.e. $b_{G,D}$ is the smallest distance between the frequency responses of G and D.
- We also define $b_{opt}(G) = \sup_D b_{G,D}$, i.e. this is the largest value over all possible linear controllers D.

We want b to be as large as possible since then the quantity in the norm signs of (15.25) will be as small as possible. (This will correspond to minimising S and making $T = 1$ as we discussed before in Section 15.4.)

Theorem 15.1 Given a nominal plant G_1, a controller D and a scalar β, then (G_1, D) is stable for all plants G_2 satisfying $\delta\nu(G_1,G_2) \leq \beta$ if $b_{G_1,D} > \beta$.

Theorem 15.2 Given a nominal plant G_1, a perturbed plant G_2 and a scalar β satisfying $\beta < \beta_{opt}$ (G_1) then (G_2, D) is stable for all controllers D satisfying $b_{G_1D} > \beta$ if $\delta\nu(G_1,G_2) < \beta$.

15.8.4 The insight provided by the ν gap metric

The three quantities

$$b_{G_1D}, b_{G_2D}, \delta_\nu(G_1, G_2)$$

obey the following triangle inequality, visualisable in Figure 15.17:

$$b_{G_1D} \geq b_{G_2D} - \delta_\nu(G_1, G_2) \tag{15.28}$$

The distance between models G_1, G_2 may be considered to be model uncertainty, and the idea can be taken further as follows.

Consider the set of process models

$$\{G : \delta_\nu(G_1, G_2) \leq \beta\} \tag{15.29}$$

Then any controller D satisfying

$$b_{G_1D} > \beta$$

will stabilise every process model in the set specified in (15.29). Figure 15.18 is a visualisation aid to accompany the above result.

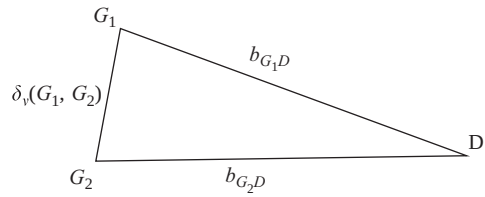


Figure 15.17 Visualisation of the triangle inequality

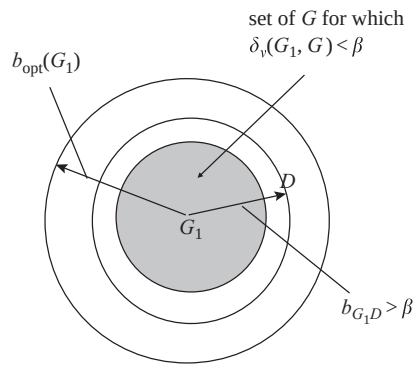


Figure 15.18 The controller D can stabilise every process whose model is within the inner circle

15.8.5 Taking into account the weights shown in Figure 15.16 to allow loop shaping

In order to reflect the performance and robustness requirements of individual designs, it will be necessary to include the weights $w_i(\omega)$ shown in Figure 15.16 into the definitions to achieve loop shaping. The general idea as outlined above will be unchanged but robustness will now need to be achieved while observing loop shaping constraints that have been built into the definitions.

15.8.6 A discussion on the two metrics δv and $b_{G,D}$

Consider two quite different industrial design scenarios:

(i) A cruise (automatic highway speed) control is being designed that must operate on a range of trucks having different engine/transmission types. Further, each truck, in service, will operate with a range of loads over a range of highway gradients.
(ii) A steel strip rolling mill rolls a variety of products of differing widths, thickness, temperatures and hardnesses and is to have an automatic thickness control system designed.

In both cases, it would be quite routine to design the necessary controllers using well-established classical techniques were it not for the large envelope of

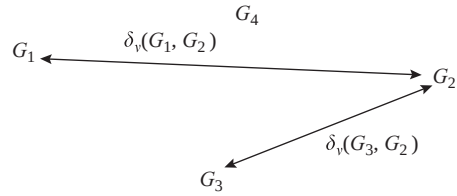


Figure 15.19 How the ν gap metric and the b metric combine to provide powerful quantitative insight into stabilisability and robust control

variability in the processes to be controlled. Almost every real industrial control task has either feature (i), where a single controller is to be designed to be fitted into a wide range of products and the hope is to avoid having to customise for each application, or feature (ii), where a single process has to produce a range of products whose varying characteristics form part of the control loop.

Suppose we were able to write a small number of transfer functions G_i that together spanned the required range of process variability. The G_i might vary in terms of parametric uncertainty or in terms of structure, or both. The ν gap metric would then allow us to plot the G_i in a visualisable plane mutually separated by distances

$$\delta_\nu(G_i, G_j)$$

as indicated in Figure 15.19.

Now encircle each of the G_i in Figure 15.19 by its own circle of radius $\beta_{opt}(G_i)$ as shown in Figure 15.20.

Each circle defines the region within which stabilising controllers D_i, for that particular G_i, certainly exist. In the illustration given here, a range of constant controllers exists, in the region marked by the starred arrow in Figure 15.20, any one of which can stabilise any of the processes G_2, G_3 and G_4. The diagram indicates that a stabilising controller may not exist for the process G_1.

The ν gap metric approach is most valuable for multivariable problems where intuitive classical loop shaping cannot be applied.

Vinnicombe (2001) is the source for all the material of this section and that reference contains a systematic and detailed exposition with examples and proofs.

15.9 Using LMI methods in control systems analysis and design

At one stage in the development of control systems design methods there was a phase during which optimisation techniques were applied intensively to a wide range of problems. Although the resulting control loops that were synthesised in that way would almost always perform as expected when tested by simulation, they tended to fail when applied to real problems. There were many discussions on the 'gap between theory and practice'. Frequently, the real reason was the excessive

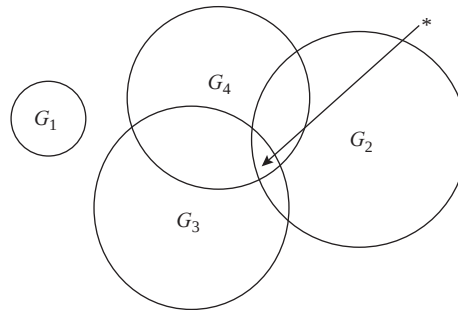


Figure 15.20 The same processes, as in Fig. 15.19, now surrounded by their circles of radius b_{opt} (G)

idealisation of the original problems necessitated by the demands of the clean mathematical lines of the available optimisation algorithms that failed to work reliably when applied to real processes.

Largely, this was because the algorithms were excessively sensitive to quite small mismatches between the process models used in control design and the processes themselves. Control engineers working in industry were well aware of the vagaries of their own processes whose behaviour could certainly not be described by constant parameter models with superimposed, well-behaved, zero-mean Gaussian disturbances. In fact, most processes tend to have some reasonably constant periods punctuated by large changes when operating conditions are changed and in addition there are superimposed short-term and long-term deterministic drifts.

At that time, it was quite impossible to even begin to pose an optimisation problem realistically taking into account the known variabilities and idiosyncracies of real processes.

That problem was partially solved using some of the early robust control techniques and more realistic design approaches slowly developed. In particular, the publication of *Linear Matrix Inequalities in System and Control Theory* by Boyd (1987) was significant.

Current, highly applicable LMI approaches allow, to quite an extent, what engineers always knew about plant variations to be fed into the design process in a highly intuitive way. To give an illustration: In the design of a control system to stabilise an electric power system network, it is now possible to reliably design a control system to be robustly stable over a wide envelope of different network configurations, different electrical loads, different generator interconnections and under certain envisaged fault conditions. All this is achieved by designating and then creating regions in design space that are realised mathematically as the intersection of a number of LMI-defined regions. Such problems, so defined, can only be solved numerically and iteratively, usually in two main stages. In the first stage, the algorithm searches to ensure that there are some solutions that satisfy the

usually three requirements that characterise the LMI problem (these are (a) an LMI that specifies the required region in design space, (b) a linear matrix equality (LME) that is a process model and (c) a cost function (J) that will be used as the scalar-valued function to be minimised during optimisation). This first stage is often called the feasibility stage. It generates a set of candidate solutions. Note that this set may be empty. The second stage, the optimisation stage, delivers the final design. It is iterative and terminates only when the cost function J has been minimised.

15.9.1 Definition of an LMI

An LMI can be concisely described by the general form

$$F(X) = F_0 + \sum_{i=1}^{m} x_i F_i > 0$$

where $X \in R^m$ is the variable whose value is sought and the m symmetric matrices $F_i = F_i^T \in R^{n \times n}$ are given. Importantly, the set $\{x: F(x) > 0\}$ is convex.

15.9.2 Solving an LMI: the feasibility stage: worked example

Computationally, LMIs are usually solved in two stages. The first stage is to establish, numerically, that *some solutions* exist that satisfy all the constraints of the problem. That is called the feasibility stage and the result is that a set r of feasible solutions is generated or it is discovered that there are no solutions – i.e. the set of feasible solutions is empty.

Below is an example of the feasibility stage for a simple problem where the feasibility may be investigated by manual calculation.

Check the feasibility of the problem; find the unknown set of X (the feasible set) that satisfy the two inequalities below, where the F_i matrices are given, or find that no such X exists (i.e. the problem is infeasible).

$$X = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} > 0$$

$$\text{with } F = F_0 + F_1 x_1 + F_2 x_2 + F_3 x_3 < 0$$

$$\text{where } F_0 = \begin{pmatrix} 2 & 3 & 2 \\ 3 & 5 & 2 \\ 2 & 2 & 4 \end{pmatrix}, \; F_1 = \begin{pmatrix} -2 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$F_2 = \begin{pmatrix} -1 & -3 & 2 \\ -3 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix}, \; F_3 = \begin{pmatrix} 0 & -2 & 0 \\ -2 & -4 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

This simple example can be solved by inspection as follows.

$F < 0$ requires that all diagonal elements satisfy $F_{ii} < 0$, by inspection, diagonal element $F_{33} = 4 + x_1$, so that we must have $x_1 < -4$.

Meanwhile, the condition $X > 0$ requires that all diagonal elements satisfy $x_{ii} > 0$ and in particular $x_1 > 0$. The contradiction indicates that no X exists that satisfies the stated requirements and the set of feasible X is empty.

Of course, had a non-empty feasible set been identified then the next stage would normally have been to search within the feasible candidates to identify a unique solution that was optimal in that it minimised a given cost function. (*Author's note: I picked up the numerical example above from the midst of a discussion forum on LMI methods, but do not have sufficient information to attribute authorship.*)

15.9.3 LMI applications to control: simple examples

A familiar stability example that is in essence due to A.M. Lyapunov in his 1892 thesis is the following.

15.9.3.1 A linear system

$$\dot{x} = Ax$$

where x is an $n \times 1$ vector, A an $n \times n$ matrix of constants is stable if and only if there is a positive definite matrix P satisfying

$$V(x) = \dot{x}^{\mathrm{T}} Px > 0, \quad \text{for } x \neq 0$$
$$x^{\mathrm{T}} PAx + x^{\mathrm{T}} A^{\mathrm{T}} Px < 0, \quad \text{for } x \neq 0$$

To prove stability, one must find some P that satisfies the two LMIs

$$P > 0, \quad PA + A^{\mathrm{T}} P < 0$$

15.9.3.2 The discrete time linear system

$$x(k+1) = Ax(k)$$
$$\text{is stable} \Leftrightarrow V(x) = \dot{x}^{\mathrm{T}} Px, \quad P > 0$$
$$\text{and} \quad V(x(k+1)) - V(x(k)) < 0$$
$$\text{or} \quad x^{\mathrm{T}}(k) A^{\mathrm{T}} PAx(k) - x^{\mathrm{T}}(k) Px(k) < 0, \quad x(k) \neq 0$$

the last inequality reduces to the LMI

$$A^{\mathrm{T}} PA - P < 0$$

So, as in the continuous case, stability of the system can be proved provided that a suitable P matrix can be found that satisfies two LMIs.

(There is a converse *instability theorem*, sometimes referred to as Chetaev's theorem, which is a mirror image of the above, allowing conditions for guaranteed instability to be determined provided that a negative definite Q matrix can be found that satisfies exactly opposite conditions to those stated above for P.)

15.9.4 Motivating example: robust pole placement using LMIs

A good example of realistic specification of an industrial problem in LMI terms is given in Rao and Sen (2000), where the task was to design a stabilising controller for a power systems network. The approach was to linearise the non-linear power systems model at a large number of different operating points and to design a controller that can work stably at every one of those operating points. The authors define a corresponding LMI region in design space, which in their case is the complex plane, by forming the intersection of several more basic LMI regions.

First, they note that a subset D of the complex plane is called an LMI region if there exist two $m \times m$ matrices α, β such that

$$D = \{z \in C | \alpha + z\beta + z^*\beta^{\mathrm{T}} < 0\}$$

Then, since the intersection of any number of LMI regions is also an LMI region and convex regions of the complex plane that are symmetric with respect to the real axis are also LMI regions, the authors are able (as shown in Figure 15.21) to define a target region in the complex plane that suits their control design application.

For example, Rao defines LMI region (1) of the complex plane by

$\sigma < -0.5$ by

$z + \bar{z} < 2(-0.5) \Rightarrow 1 + z + \bar{z} < 0$

here $\alpha_1 = \beta_1 = 1$

and LMI region (2), which is a conic section that makes an angle θ with the imaginary axis, is defined by

$$\begin{pmatrix} \cos\theta(z+\bar{z}) & -\sin\theta(z-\bar{z}) \\ \sin\theta(z-\bar{z}) & \cos\theta(z+\bar{z}) \end{pmatrix} < 0$$

$$\text{or } \alpha_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\beta_2 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Finally, setting $\theta = 0.1$, corresponding to a damping factor of $\zeta = 10\%$, and combining the α and β values lead to

$$\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.995 & -0.1 \\ 0 & 0.1 & 0.995 \end{pmatrix}$$

and the required LMI region D is defined.

The paper (Rao and Sen, 2000) then goes on to plot simulation results of possible state feedback schemes in which trial configurations of power system stabilisers (used to minimise low-frequency network oscillations) were tested under different system loading conditions. The results show how the robustness (as quantified by the closed loop pole positions in the LMI region D described above) can be quantified for different feedback control arrangements.

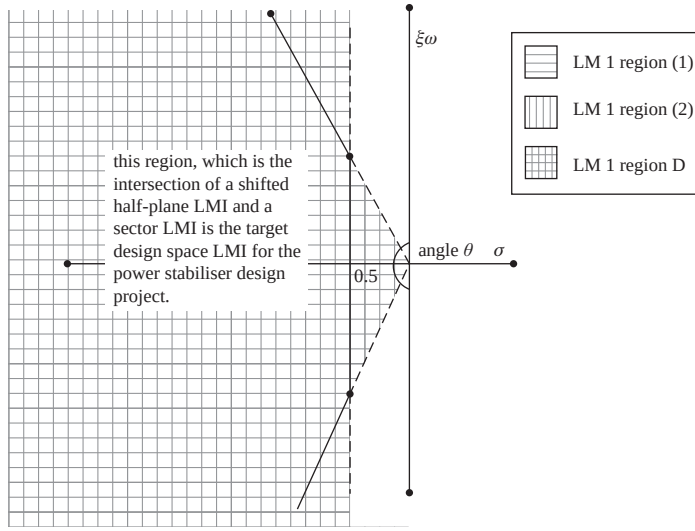


Figure 15.21 A desired LMI region in the complex plane where the poles of a synthesised system will be placed in two stages: First, a feasibility stage will show that some solutions can be produced whose poles lie in the designated LMI space. Second, an optimisation stage will search for the best among the feasible solutions

15.10 An outline of how H_∞ design works and how its practicality can be usefully extended through μ synthesis

Figure 15.22 shows the standard configuration that both Matlab and Scilab require to be presented to H_∞ and many other controller design programmes.

In the figure, the fixed parameter process model $P = \{A, B, C, D\}$ is decomposed in an obvious way with the B, C, D matrices and the input–ouput signals being split according to whether they participate in the closed loop through the to-be-designed LTI (linear time invariant) controller K or are part of the external connection.

The H_∞ design algorithm *hinfsyn*, given the populated Figure 15.22, will, by choice of parameters in the controller K, minimise

$$\max_\omega \left(\frac{||y_2||_2}{||u_2||_2} \right)$$

which is the H_∞ norm, to be visualised as the highest possible 'gain', over all frequencies, of the ratio y_2/u_2, where $u_2(s) = Ky_2(s)$.

In a real application, the user will input minimum acceptable performance requirements that will need to be combined into the overall minimisation computation. This is where human interaction with the programme comes in; asking too

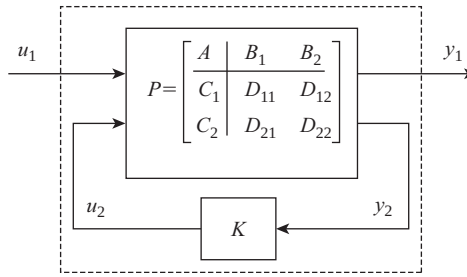


Figure 15.22 The standard configuration for H_∞ robust controller design

much in terms of performance may well result in the routine failing to find any solution that can meet both performance and stability requirements. In such cases, the designer will possibly relax the performance specification or the requested stability margin or revisit the modelling assumptions.

15.10.1 The μ-synthesis method

This method can be considered as a further useful development from the H_∞ method that allows for process parameter uncertainty to be more specifically taken into account.

In the μ-synthesis approach, the diagram Figure 15.22 is still broadly applicable but additional input and output weighting factors are incorporated in such a way that dimensionless input–output signals of unity peak magnitude are generated.

The μ-synthesis programme then searches for a controller K that yields a maximum singular value σ satisfying

$$\overline{\sigma}_{all\ \omega}\left(\frac{||y(s)||}{||u(s)||}\right) \leq 1$$

From a user's point of view, this method has the great advantage that physically understandable uncertainty about process model parameters can be directly encoded in everyday process operator's language. For example, suppose that the process model contains parameters a, b, c with nominal values $a = 1$, $b = 10$, $c = 50$, then the μ-synthesis programme can be run with those values: let us suppose that such a programme run delivers a solution for K and a value of $\sigma = 0.9$. Clearly, we have a candidate solution that satisfies our requirements. However, we now learn that the process parameters are subject to the following uncertainties:

$$a = 1 \pm 0.1,\ b = 10 \pm 0.5,\ c = 50 \pm 1$$

The Matlab μ-synthesis programme allows such uncertainty intervals to be inserted and if the programme is re-run with such uncertainties included, it is certain that the value of σ will have increased and may no longer satisfy the relation $\overline{\sigma} < 1$.

The purpose of this illustration is to demonstrate how the μ-synthesis approach has such a long practical reach.

The Matlab μ-synthesis programme can be called by the command ***dksyn***, because it makes use of a so-called *D-K* iteration in which candidate LTI controllers K are iterated with candidate frequency-dependent scaling matrices $D(\omega)$, until a solution is found that may or may not satisfy the required inequality $\overline{\sigma} < 1$.

For more information on the theoretical background, see the Matlab Robust Control Toolbox literature, which also contains further lists of references.

For an application that uses this technique, see Chen *et al.* (2011).

15.11 Robustness or adaptivity?

A robust controller is designed to control all processes having transfer functions, loosely speaking, in the range $G + \Delta G$ where ΔG represents either

(i) a bound on modelling uncertainty or
(ii) an estimate of the envelope of variability for the process over different expected situations.

Where ΔG, the region of process uncertainty, is large, the performance with any fixed robust controller may be inadequate for the application. In such a case, there may be an advantage in introducing a degree of adaptivity into the controller, allowing it, as far as possible, to track the parameters of the actual process, instead of having to allow a priori for the possible spread of parameters.

The decision on whether to use robust design, some type of adaptive control or a combination of the two will need to be made on a case-by-case basis, taking into account the rate of change of process characteristics and the identifiability of the process parameters. (See Section 16.5.1 for a further discussion.)

15A A hierarchy of spaces

Figure 15.23 shows how spaces are axiomatically defined with increasing structure as one passes down the diagram starting from topological spaces with few properties except connectedness, down through metric and normed spaces possessing measures of size and distance, to the Lebesgue and Hardy spaces that give theoretical underpinning to much of control theory.

Lebesgue spaces $L^p[a, b]$

Lebesgue spaces L^p $[a, b]$ (named after Henri Lebesgue (1875–1941) who developed the modern rigorous theory of integration based on a foundation of his pioneering work on measure theory) are defined as spaces of functions f where the integral exists.

$$\left(\int_a^b |f(t)|^p\right)^{1/p}, \quad p \in [1, \infty]$$

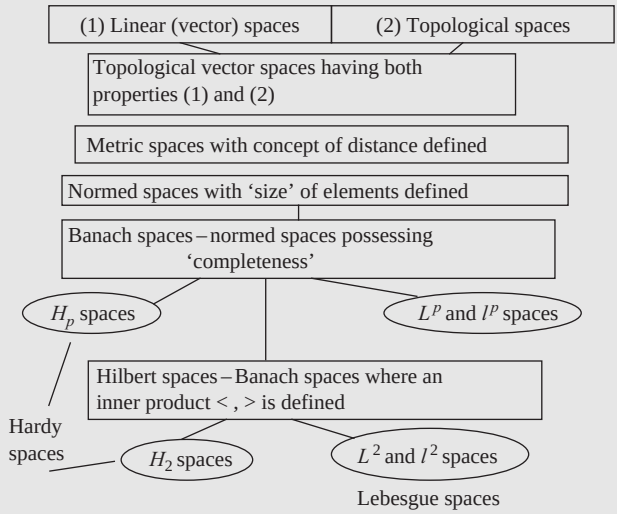


Figure 15.23 A hierarchy of spaces showing increasing structure as the diagram progresses downwards

The L^p spaces are linear (vector) spaces since the sum of two integrable functions is again integrable and the scalar multiple of an integrable function is again integrable. Note also that in L^p spaces we are always dealing with equivalence classes of function rather than with individual functions. This arises because functions that differ only at isolated points (more formally, functions that differ only on a set of measure zero) are identical from an L^p point of view.

Sequence spaces l^p

Let X be a set of sequences $\{x_i\}$ of real numbers. Let every such sequence satisfy

$$\left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} \leq m < \infty$$

where p is a real number

$$p \in [1, \infty)$$

Then m is a norm for X and X is called an $\boldsymbol{l^p}$ space. When $p = \infty$, we define

$$||x||_\infty = \sup(|x_i|)$$

Inclusion relations between spaces

Let P be the space of all polynomials, C_n be the space of all n times differentiable functions, C be the space of all continuous functions and let

$1 < p < q < \infty$. Then, assuming that all the functions are defined on the same finite interval

$$P \subset C^\infty \subset C^1 \subset C \subset L^\infty \subset L^q \subset L^p \subset L^1$$

Let $\mathcal{C}$ be the set of all convergent sequences, $\mathcal{C}_0$ be the set of all sequences convergent to zero, and let $1 < p < q < \infty$. Then the following inclusion relations apply amongst the sequence spaces:

$$l^1 \subset l^p \subset l^q \subset \mathcal{C}_0 \subset \mathcal{C} \subset l^\infty$$

The norm of a linear mapping *T*

The norm of a linear mapping T is usually defined in terms of a ratio of L^2 norms on the domain and range spaces.

Hardy spaces

Hardy spaces have become increasingly important in control theory since about 1985. The foundations of these spaces and their naming in 1923 in honour of the Cambridge mathematician G.H. Hardy (1877–1947) is due to the Hungarian analyst F. Riesz (1880–1956), who was one of the founders of functional analysis. Hardy spaces are important in harmonic analysis, power series, operator theory and random processes as well as in control theory.

The space H_∞

H_∞ is a member of the family of **Hardy spaces** (H_p, $p > 0$). It is the Banach space of all complex-valued functions of a complex variable that are analytic and bounded in the right half plane where

$$\text{Re } s \geq 0$$

Such functions have the norm

$$||f||_\infty = \sup_{\text{Re } s > 0} |f(s)|$$

and by Fatou's theorem, which says that these functions can be defined by their boundary values,

$$||f||_\infty = \operatorname*{ess\,sup}_{\omega} |f(j\omega)|$$

See Duren (2000) for the underpinning theory of H_p spaces.

A note on notation

There appears to be a rough consensus that Lebesgue spaces are denoted L^p spaces (p being superscript), whereas Hardy spaces are denoted H_p (p being subscript). I have followed this convention.

15B Model reduction to avoid overcomplexity in synthesised controllers

When developing a model from logged process data, there is always a problem of deciding which aspects of the draft model are significant and relevant, which aspects could be relevant on another occasion for a different application on the same plant and which aspects can be classified as noise on this occasion.

Many of the more sophisticated methods of control design will synthesise a controller of an order as great or greater than the model with which they are provided and some other types of controller include an internal process model within the algorithm that will be used eventually online; therefore, process models for controller design should have the lowest order that still correctly mirrors all significant process behaviour for the particular application that is envisaged.

There are many possible approaches to model reduction, not all of them well documented but often based on modelling common sense. For instance, if the model being developed is required specifically to discover the causes of a particular behaviour in a batch process, the approach will normally be to fit a model to each of n separate batches and use obvious statistical consistency checks to exonerate some possible causes and to order the magnitudes of the remainder.

One of the most favoured systematic methods for model reduction of state space models is based on ordering the Hankel singular values of the model's transfer matrix $G(s)$. Just as the usual singular values of a matrix A can define the rank of a matrix A, so in a roughly parallel way, the Hankel singular values of a transfer matrix $G(s)$ can help to define the significance of states in the process represented by $G(s)$. (See interlude 11C for more explanation.) What the above statement means in practice is that the magnitude of the Hankel singular value of state m in Figure 15.24 is a good indicator of the relative significance of state m in input–ouput terms; meaning that states with small Hankel singular values can probably (but see below) safely be discarded from the model.

Figure 15.24 shows the normalised singular values of a 20th-order model G, as obtained by fitting to logged process data. Inspection shows that, depending on the application, possible reduced models of order 12, 7, 5 and 3 might be investigated. The way forward is first to plot a Bode diagram of the 20th-order model and from that plot and a knowledge of the dynamic aims of the proposed application decide on the frequency region where reduced-order model and 20th-order model must match fairly closely. Fortunately, in many processes, it happens that most of the states with low Hankel singular values have their influence at frequencies that are outside the range of significance.

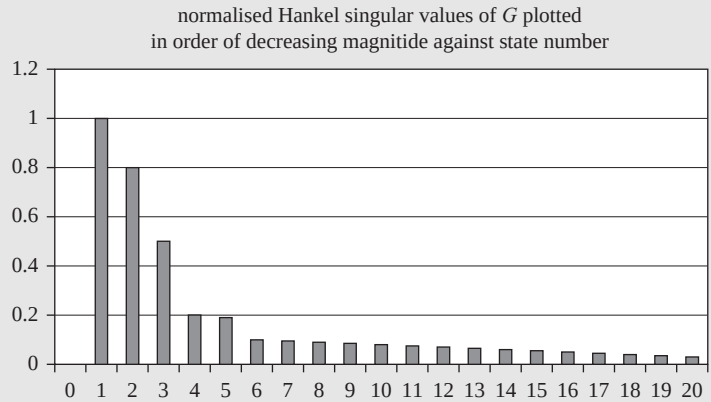


Figure 15.24 The normalised singular values of a 20th-order model G, as obtained by fitting to logged process data

See Obinata and Anderson (2000) for further details.

Chapter 16

A miscellany of control techniques

(Including neural networks, fuzzy logic and genetic algorithms; control switching, gain scheduling, adaptive and learning techniques; intelligent systems, agent-based and co-operative systems.)

16.1 Introduction

This chapter describes a selection of what are sometimes referred to as artificial intelligence (AI) techniques (in fact these methods are, in general, empirically/numerically based rather than being analytically/theoretically based like the bulk of conventional control theory).

Neural networks are sets of interconnected artificial neurons that, very simplistically, imitate some of the logical functioning of the brain. After training, they can represent any algebraic non-linearity. They have to be trained by being presented with sufficient examples of the input–output behaviour that is desired, so to a large extent they can only represent existing data-generating phenomena by empirical equivalents.

Fuzzy logic emulates the reliable but approximate reasoning of humans, who, it is said, distinguish only six or seven different levels of any variable during decision making. Fuzzy logic algorithms can represent this style of reasoning by easily understood curves that are ideal for implementing those many control systems that are based on 'having a feel' or on 'rules of thumb' rather than on equations.

Genetic algorithms and genetic programming are powerful evolutionary search methods that can search for structures as well as numerical parameters. These qualities allow the methods to synthesise solutions to a wide variety of problems. The approaches rely heavily on imitating the methods of animal/human reproduction followed by natural selection. Because the methods can search amongst many alternative structures, they can also be regarded as design or synthesis methods.

Learning systems aim to emulate the human learning-by-experience mechanism so that a system can potentially learn to perform a task with increasing efficiency over time using an iterative algorithm.

Intelligent machines and machine intelligence offer future prospects for creating systems with ever-increasing autonomy and reasoning ability. Agent-based systems and co-operative systems are two developing areas of research that aim to

allow possibly large numbers of collaborating entities to achieve significant results as the sums of their efforts.

16.2 Artificial neural networks (ANN)

16.2.1 Motivation

From a control point of view, a neural network can be regarded principally as a non-linear input–output black box that can emulate a process, a controller, a state estimator or a classifier (Figures 16.1 and 16.2). Neural nets contain coefficients called 'weights' (Figure 16.3). They need to be taught by being presented with numerical examples (that represent the desired behaviour) while the weights are modified by a training algorithm until the neural net performs as closely to the examples as possible.

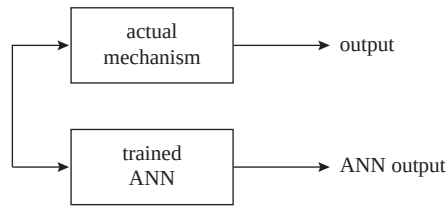
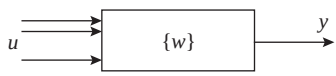


Figure 16.1 Basic abilities of neural nets: after being trained with a sufficient number of accurate examples, they can emulate any non-dynamic, non-linear mechanism

Figure 16.2 Basic abilities of neural nets: after being trained with a sufficient number of accurate examples, they can act as classifiers



A neural network has a memory {*w*} of 'weights' that are learned during training.

A neural network can be a process model, an inverse model, a controller, an estimator, a classifier or a filter.

Figure 16.3 Basic abilities of neural nets: the choice of weights w *determines the function that is emulated*

16.2.2 A basic building block: the neuron

A neural network is made by interconnecting a number of neurons (referred to equivalently as perceptrons, nodes or processing elements). Figure 16.4 shows a single neuron. It receives n inputs x_i, each x input being multiplied by a weight w_i. The neuron sums the weighted inputs, adds in a bias term b and then processes the sum through a function f to produce a scalar output y, given by the equation

$$y = f\left(\sum_{i=1}^{n} x_i w_i + b\right) \tag{16.1}$$

The function f is the choice of the user but the characteristics of the sigmoid function

$$f(x) = \frac{1}{1 + e^{-x}} \tag{16.2}$$

make it the most widely applied for general emulation purposes.

Training of the neuron implies fixing numerical values for the weights w and the bias b so that the neuron behaves in a desired way.

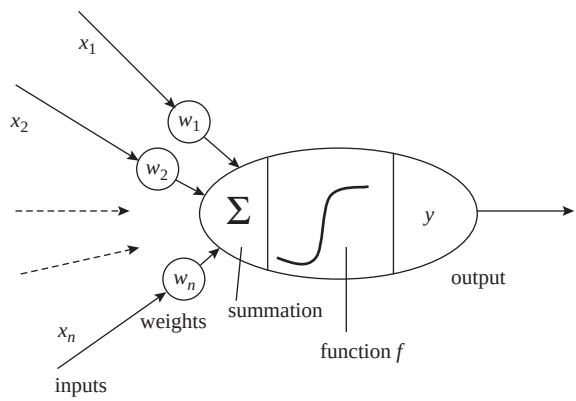


Figure 16.4 Architecture of a typical neuron

16.2.3 Simple properties of a neuron demonstrated in the two-dimensional real plane

For this illustration, we set $n = 2$ and $f = 1$. Now, if we set $y = 0$, the equation of a straight line results as

$$x_2 = -\frac{w_1}{w_2} x_1 - \frac{b}{w_2} \tag{16.3}$$

shown in Figures 16.5 and 16.6 for two different values of b.

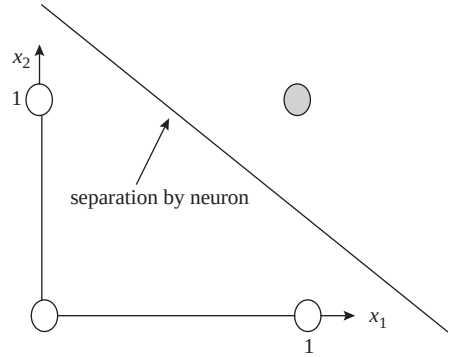


Figure 16.5 Realisation of x_1 AND x_2

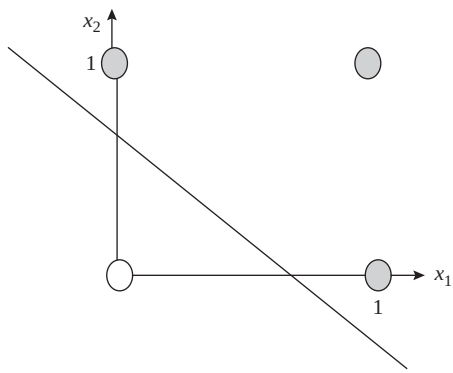


Figure 16.6 Realisation of x_1 OR x_2

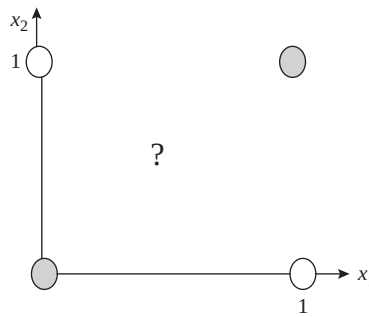


Figure 16.7 The question mark (?) indicates an important generic shortcoming of the neuron as a classifier: It is unable to generate any single line that will separate the points and realise the XOR function

It is clear from the figures that the single neuron divides the plane into two regions and can work like an AND or an OR gate, according to the value given to the bias term b. It is also clear, Figure 16.7, that no single line can separate the

points $(-1, -1)$, $(1, 1)$ from the points $(1, -1)$, $(-1, 1)$ as is required by the exclusive OR (XOR) function.

One solution for mechanising the XOR function might be to use two neurons to generate two separating lines, and then to feed the output of the two neurons into a third combining neuron to form a region. This leads to the idea that more than one layer of neurons will be needed to allow wider classes of functions to be emulated.

We shall return to the topic of multilayer networks shortly but first we consider the case of a single neuron with n inputs.

16.2.3.1 Properties of a single neuron with *n* inputs

A neuron with n inputs describes a hyperplane that separates $\mathbb{R}_n$ into two disjoint regions, say A and B. The plane with normal $v \in \mathbb{R}_n$ has the equation

$$\langle x, v \rangle = b$$

and this plane is offset by the distance b from the parallel plane

$$\langle x, v \rangle = 0$$

that passes through the origin.

A neuron with weights $w \in \mathbb{R}_n$ and bias $b \in \mathbb{R}_n$ assigns any $x \in \mathbb{R}_n$ to region A or B using the rule:

$$\langle x, w \rangle > b \Rightarrow 0\ x \in A$$
$$\langle x, w \rangle < b \Rightarrow 0\ x \in B$$

If the convex hulls of the sets A and B are disjoint then some hyperplane generated by the neuron can give perfect separation of the points into their correct categories.

If the convex hulls of the sets A and B intersect, then no hyperplane can separate the points perfectly and the best one can do is to choose the plane that misclassifies the least number of points.

16.2.4 Multilayer networks

Three layers of interconnected neurons are said to be sufficient to emulate any desired non-dynamic function. The most widely used neural network is perhaps the so-called multilayer perceptron (MLP) (Figure 16.8). An MLP usually has a three-layer architecture with input, hidden and output layers. The number of neurons in each layer and the types of functions embedded in each neuron are chosen by the designer of the network to match the application.

16.2.5 Neural network training

Neural network training is the activity of fixing the weights w and the bias terms b throughout the network until the behaviour obtained achieves some given performance objective. The most used training algorithm is back-propagation.

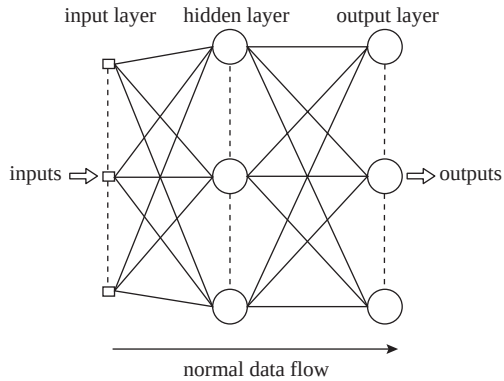


Figure 16.8 A multilayer neural network containing six neurons

This works, in principle, as follows. Training examples in the form of input–output data sets (x, y) are presented to the neural network whose output estimates y are recorded. After presentations of k such data sets, we shall be in possession of the information $(x_j, y_j, \hat{y}^j, j = 1, \ldots, k)$ and can form the error sum

$$J = \sum (y^j - \hat{y}^j)^2 \tag{16.4}$$

whose minimisation will be the training aim.

Where the neural network has only one layer, back-propagation consists only of adjusting each weight according to the algorithm

$$\Delta w_i = \frac{\partial J}{\partial w_i} \Delta J \tag{16.5}$$

where the partial derivative will only exist if the function f in each neuron is itself differentiable, such as is the case when f is the sigmoidal function.

In multilayered networks, the same principle applies with (16.5) now having the characteristic that adjustments to weights in early layers can be found only once the later layer corrections have been calculated; hence, the name back-propagation.

In practice, the training of a large neural net on industrial data needs to follow a procedure such as the following. The available input data set Q is divided into three subsets, say A, B, C. The network is trained to fit the training set A, with periodic checks to determine the goodness of fit of the partially trained network against verification data set B. The idea of this procedure is that training can be continued too long ('overtraining') such that the network 'learns' the data set A, noise and all, in great detail and no longer captures the underlying function so well as in the earlier stages of learning. By using the set B, the point where overtraining is imminent can be detected, the training stops and the performance against unseen data C can be checked (Figure 16.9).

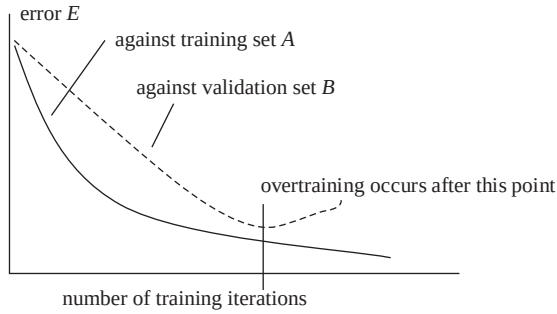


Figure 16.9 Illustrating the phenomenon of overtraining

(The problem of overtraining, or overfitting, is not confined to neural net applications and occurs whenever high-order models are fitted to noisy or batch-to-batch varying data from a process of lower order. However, because neural nets tend to be of high algebraic order (a large number of weights to be trained), the overtraining problem is more severe than in classical modelling using, for instance, differential equations.)

16.2.6 Neural network architectures to represent dynamic processes

All the neural networks we have discussed so far have been non-dynamic. That is, input information is immediately processed and appears without storage or delay at the output. In contrast, a dynamic process has internal storage and a transient response.

To see this, look at what happens to a dynamic system that receives a step input (Figure 16.10). Although the system receives a constant input of unit magnitude,

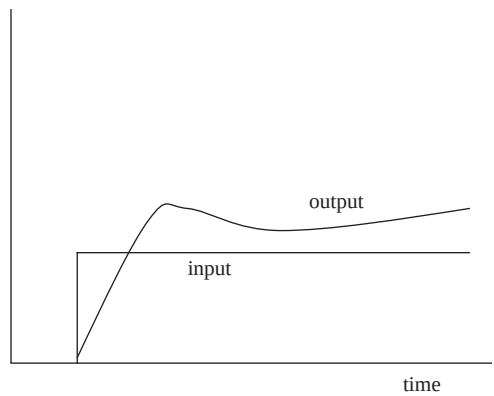


Figure 16.10 A neural net with some sort of dynamic feature is clearly needed to learn this sort of input–output behaviour (in a normal non-dynamic net, the same input will always produce the same output)

the corresponding output, as shown in the figure, depends on the time. This feature makes neural network training more difficult than simply choosing weights to represent a time-invariant relationship.

16.2.6.1 Three ways to make neural networks dynamic

(1) Make the network recursive (Figure 16.11). From the figure,

$$(1 + Pz^{-1})y(z) = Pu(z)$$

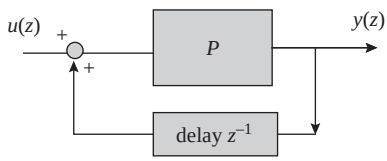
$$\frac{y(z)}{u(z)} = P(1 + Pz^{-1}) = \frac{Pz}{P + z}$$

(2) Provide the system with delayed inputs alongside normal inputs yielding (Figure 16.12)

$$\frac{y(z)}{u(z)} = \frac{P(z + 1)}{z}$$

(3) Add an integrator to the network yielding (Figure 16.13)

$$\frac{y(z)}{u(z)} = \frac{P}{z}$$

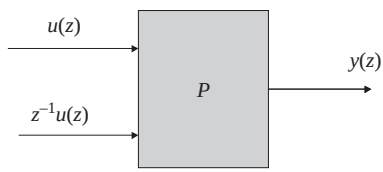


The configuration has the first-order dynamic equation

$y(z)/u(z) = Pz/(P+z)$

which has a first-order dynamic.

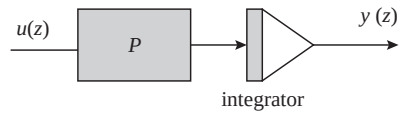
Figure 16.11 Neural element P, made dynamic by feedback (recursive network)



This configuration has the first-order dynamic equation

$y(z)/u(z) = P(z+1)/z$

Figure 16.12 Neural element made dynamic by delayed inputs alongside normal inputs



This configuration has the first-order dynamic equation

$$y(z) = z^{-1}Pu(z)$$

$$y(z)/u(z) = P/z$$

Figure 16.13 Neural element made dynamic by connection of a separate integrator

The simple derivations for networks (2) and (3) are similar to that shown for (1) above and are omitted.

Most important industrial processes are non-linear and dynamic. If the dynamics are modelled by a conventional network and the non-linear part by a neural net, excellent results can often be obtained. However, in such a configuration, network training can be difficult since differentiation of the industrial data, with loss of information, may be required if back-propagation approaches are to be used.

The ease of application of neural nets and the speed with which tolerable results are delivered has caused many users to neglect to study the problem properly and to neglect a careful pre-treatment of the data. The two omissions combined can lead to quick and cheap empirical solutions that will be expensive in the longer term. A very successful solution to this problem of excessive empiricism is to

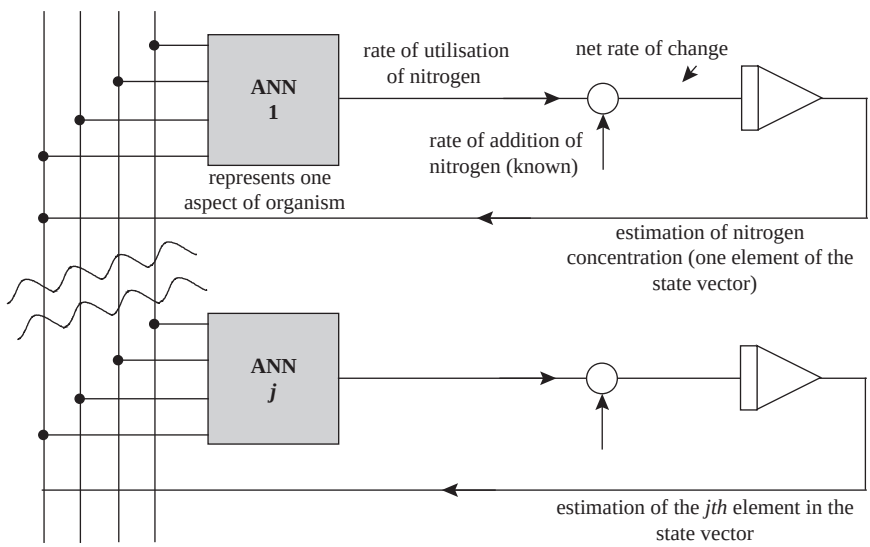


Figure 16.14 How neural nets can be embedded within known dynamics to produce a transparent and mathematically sound state estimator (the example is from a large fermentation process)

embed small scale neural nets within a conventional model of the known dynamics of a process to obtain a state variable structure as shown in Figure 16.14.

Such a structure is both mathematically sound as well as transparent (rather than black box).

16.2.7 Using neural net-based self-organising maps for data reduction and clustering

Self-organising maps (SOMs), particularly using the Kohonen approach, find application in clustering high-dimensional data by unsupervised mapping onto a space of reduced dimension. Typically, several hundred input 'patterns' will be input to the SOM that will self-learn a small number of feature patterns at the centre of the classifying clusters.

A SOM, used in this way, can be regarded loosely as a neural net-based non-linear equivalent of a principal components analyser (PCA).

16.2.8 Upcoming rivals to neural networks? Support vector machines and adaptive logic networks

Support vector machines (SVMs) (Schölkopf *et al.*, 1998, 1999; Cristianini and Shawe-Taylor, 2000) work by mapping data into high-dimensional feature space and in that space, linear functions are fitted to the features.

Adaptive logic networks (ALNs) use a growing self-organising tree of piecemeal linear functions or hyperplanes.

The proponents of these two approaches claim that they are faster and more transparent than neural networks, that they have global minima and that they also allow the inclusion of domain knowledge during the modelling process. Under some conditions, ALNs can be reversed so that the output becomes the input. This ability to invert a learned function can have great utility in allowing analysis to be turned into synthesis.

ALNs are trained in a similar way to neural nets but they can also be trained by reinforcement learning in which only rough fuzzy feedback such as 'good' or 'poor' is provided by the supervisor.

16.2.9 Neural nets: summary

- Very simple idea of interconnected neurons that can emulate any function for which numerical examples are available.
- Some theoretical support from Weierstrass' theorem – any continuous function may be approximated arbitrarily closely by a polynomial.
- An ANN is a ready-made modular polynomial with an effective back-propagation method of parameter fitting.
- Not so good as a well custom-constructed non-linear dynamic model but the effort required is very much less.

16.3 Fuzzy set theory and fuzzy logic

16.3.1 Introduction and motivation

To the extent that mathematics is exact it does not apply to the real world; to the extent that it applies to the real world it is not exact

Precision is not truth

Precision and relevance can become almost totally mutually exclusive characteristics

These quotations (from Einstein, Matisse and Zadeh) confirm our experiences that everyday situations are in general too imprecise to be dealt with satisfactorily by mathematical tools.

These three quotations appear to argue in favour of imprecise but reliable human reasoning and action taking. Our everyday observation is that small children rapidly learn to catch a ball, make a swing go really high, ride a cycle or roller skate, all based on 'acquiring a feel'. The attraction of controllers that might acquire a feel, instead of requiring to be based around a complex quantitative dynamic model, is obvious; controllers based on fuzzy logic go some way towards encoding the human ability to 'acquire a feel'.

Normal set theory and normal logic are characterised by formalised precision. For instance, once set A has been defined then every element in the universe of discourse belongs either to A or to the complement of A (Figure 16.15). Similarly, every statement in logic produces a statement of either 'true' or 'false' with no possibility of 'maybe'.

In contrast, fuzzy set theory is characterised by imprecision, and since human reasoning is based on approximations, here lies the attraction of fuzzy sets. We can, for instance, define the set of all 'fierce dogs' or the 'set of all bad restaurants' it being understood that there will be different degrees of 'fierceness' and 'badness', respectively.

The idea of a stepped grey scale (Figure 16.16) comes to mind to quantify membership of a fuzzy set. Considering again the set of all fierce dogs, normal set

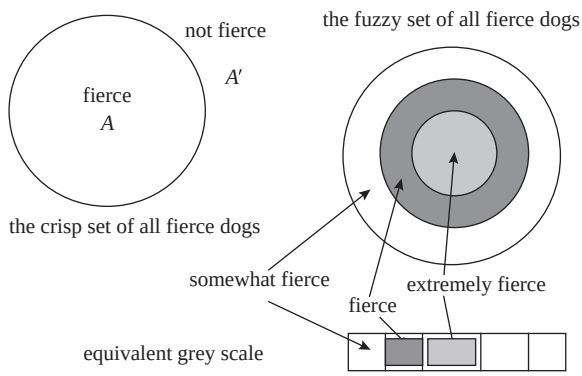


Figure 16.15 Crisp and fuzzy sets

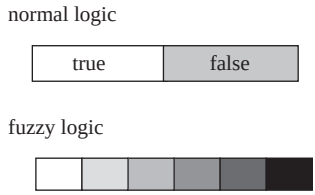


Figure 16.16 Normal and fuzzy logic

theory would have a crisp 0-1 classification into fierce and non-fierce. Fuzzy set theory would have some well-defined transition from most fierce to not fierce, leading to the concept of a broad fuzzy set boundary, and the idea of degrees of set membership.

It is clear that a fuzzy set can contain more useful knowledge for everyday decision making than can an equivalent crisp set. The attraction of fuzzy logic/fuzzy set theory is that it allows common sense encoding of different levels of intensity and it also allows for the outputting of different levels of activity, leading straightaway to the idea of a fuzzy logic controller.

In particular, fuzzy logic often allows the simple mechanisation of the control actions of a human operator. Mamdami (1976) was the first to publish reports of fuzzy control of a model steam engine while the first successful applications of fuzzy control in industry was to a cement kiln where operators look at many subjective quantities and then adjust a few process variables. Fuzzy logic proved ideal for codifying the operators' rather ill-defined but reliable control actions at the Danish plant of L.A. Schmidth (Holmblad and Ostergaard, 1982).

16.3.1.1 A simple illustration of how a crude rule of thumb can be encoded to produce an easily implementable control algorithm

Imagine a situation where a furnace has the rule of thumb for control as follows:

- If the indicated temperature is LOW (90°C or less), then set the fuel valve (FV) to 100.
- If the indicated temperature is OK (near to 100°C), then set the FV to 10 (this setting having been found to just offset the losses occurring at 100°C).
- If the indicated temperature is HIGH (110°C or higher), then set the FV to 2. (Let us agree that it is not allowable to shut off the fuel completely and that this is the minimum allowable setting.)

Our chosen fuzzy control algorithm simply interpolates linearly in the above rule of thumb (Figure 16.17) to give the rule

$$\theta \leq 90^\circ \Rightarrow \text{FV} = 100$$

$$90^\circ < \theta \leq 100^\circ \Rightarrow \text{FV} = 100 - 90\frac{(\theta - 90)}{10}$$

$$100^\circ < \theta \leq 110^\circ \Rightarrow \text{FV} = 10 - 8\frac{(\theta - 100)}{10}$$

$$\theta > 110^\circ \Rightarrow \text{FV} = 2$$

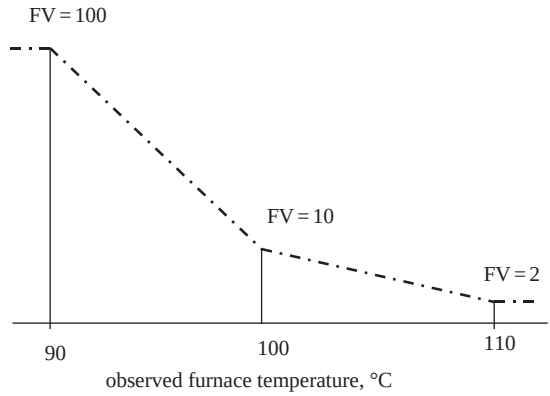


Figure 16.17 Actions required: solid line – rule of thumb; dotted line – fuzzy logic interpolating curve

In use, the algorithm would be run every T seconds, with T being chosen to suit the dynamics. The value of FV would be held constant between calculations.

16.3.1.2 Fuzzy control can deal with very complex and ill-defined problems that defy mathematical analysis

In a collaborative project between University of Westminster and a UK cement manufacturer, there were around 40 measured or observed variables as inputs to a fuzzy control algorithm, but only some 3 or 4 variables to be controlled. Fuzzy logic techniques allow such problems to be visualised and driven graphically so that the many interacting and even contradictory laws can be weighted (based on operators' advice) and then combined to form a number of required action shapes. The actions to be implemented at each time step are then found, typically, by finding the centres of areas of those required action shapes.

16.3.2 Some characteristics of fuzzy logic

- Imprecise rules of thumb may easily be encoded.
- Simple structures that parallel human reasoning result.
- The overall operation of a fuzzy logic control can be visualised graphically.
- Using fuzzy logic it is easy and practicable to engineer custom solutions to practical problems using solutions that can successfully encode and then interpolate in operator wisdom and operator feel.
- Fuzzy logic allows mathematics to change its character to emulate the reliable but approximate decision-making methods that humans have evolved so successfully over the centuries.
- Disadvantages of control based on fuzzy logic.
- Many concepts/tools of conventional control are not easily available (such as frequency response, stability margin).
- Because of the above, fuzzy control solutions have to be checked out empirically over a range of scenarios, rather than being guaranteed mathematically.

16.3.3 References: early pioneering work

Holmblad L.P., Ostergaard J.J. 'Control of a cement kiln by fuzzy logic' in Gupta M.M., Sanchez E. (eds.). *Fuzzy Information and Decision Processes* (North Holland, Amsterdam, 1982), pp. 389–99. (This paper surveys the application of fuzzy logic by F.L. Smidth & Co. (FLS) for control of rotary cement kilns. The presentation is given in retrospect, starting in 1974 when FLS heard about fuzzy logic for the first time. The most important milestones are presented, with special emphasis on the role of fuzzy logic.)

Mamdami E.H. 'Applications of fuzzy algorithms for control of simple dynamic plant'. *Proceedings of the IEEE*. 1976;**121**:1585–88

Zadeh L.A. 'A rationale for fuzzy control'. *Journal of Dynamic Systems, Measurement and Control*. 1972;**94**(Series G):3–4

16.4 Genetic algorithms

16.4.1 Basic ideas

Populations of living organisms have powerful abilities to evolve and to adapt, guided by actual experiences (survival of the most fit for purpose). Genetic algorithms (GAs) imitate natural evolution and natural selection to find solutions to a wide variety of search problems. Natural evolution has a number of features that can possibly be transferred to artificial GAs. These are as follows:

(1) A blueprint for a new organism, being a chromosome encoding future characterisation as a string of symbols.
(2) In many organisms, a sexual generation mechanism in which two chromosomes from the two parents line up and make a linear exchange of genes from a randomly selected point onward. This mechanism is called crossover.
(3) A (possibly infrequent but important) mutation mechanism that ensures that entirely new regions of the search space are occasionally accessed.
(4) A survival of the 'most fit for purpose' strategy. In nature, this strategy is administered by the ability of an organism to survive and even thrive in a competitive environment, at least to a point where it has parented its own offspring.

16.4.2 Artificial GAs

(1) Every potential solution to a search problem to be solved by a GA approach must somehow be encoded as a string of (say) binary symbols in such a way that all allowable strings are possible solutions (Figure 16.18).
(2) Crossover and mutation strategies (Figure 16.19) exist, imitating the natural mechanisms described above.
(3) A fitness function is used to linearly order any set of possible candidate solutions.

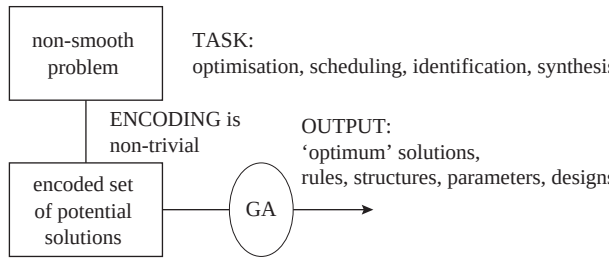


Figure 16.18 Genetic algorithms (GAs) are general purpose optimisation algorithms

The problem to be solved can be considered amongst the class of hill-climbing problems where visualisation is in the form of a landscape in which we seek the highest point, equivalent to the point of highest elevation, as measured by the fitness function.

In hill-climbing, smooth landscapes with single (i.e. unimodal) maxima are relatively easy to solve, whereas noisy landscapes with multiple maxima confuse and delay the algorithm.

All search methods progress more slowly when the problem is non-linear, non-smooth, noisy and with multiple maxima. However, the GA, properly set up, has shown itself to be one of the most effective general search methods for such difficult problems.

To understand the particular effectiveness of the GA approach, consider a mountainous landscape that represents the search problem with the task being to find the point of highest elevation. The population of candidate solutions is initially randomly and more or less uniformly distributed across the search space.

However, as successive generations evolve, the 'net' of candidate solutions becomes ever more closely meshed near to possible solutions and correspondingly

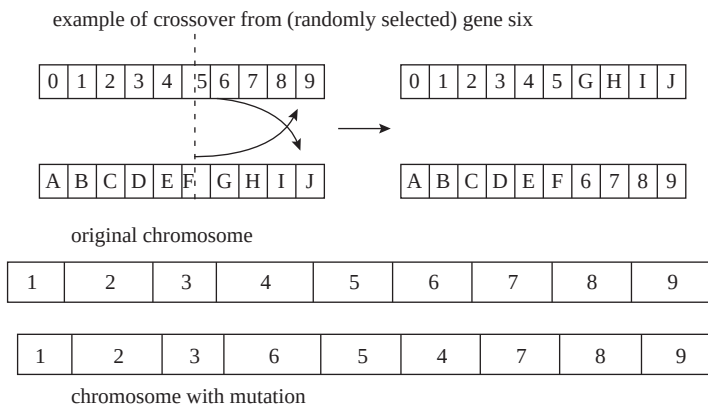


Figure 16.19 Illustration of the crossover and mutation mechanisms

How a chromosome can be set up to encode
(a) qualitative structural information (type of architecture)
(b) quantitative structural information (numbers of nodes/layers, etc.)
(c) numerical values of parameters, for instance, for a neural network model of a process

Figure 16.20 Because of an ability to search among widely differing disparate structures, a GA can be considered to be a design and synthesis tool

sparse far away. Thus, the search is parallel with statistically increasing probability of search near to likely solutions and although fragmented summits (spiky noise) necessarily delay any method of solution, the GA's lack of direct reliance on seeking directions (which are badly affected by local noise) puts it at an advantage.

Thus, GAs are able to concentrate most of their attention on the most promising parts of the problem space.

16.4.3 GAs as design tools

In considering how a set of solution strings develops towards the required solution, it becomes evident that the spatial location of information within the chromosome may be important. Consider using a GA to choose the architecture and train the weights of a neural network to model a dynamic batch process for which input–output data are available. In such a case, one segment of the chromosome could represent structure or type of architecture, another segment, numbers of layers and types of embedded functions, while the final layer could represent the numerical parameters that need to be estimated (Figure 16.20).

It is clear that GAs with their ability to choose between alternative structures and, as it has been shown by Koza *et al.* (1999), their ability to synthesise novel structures and novel solutions, make them very powerful tools.

16.4.4 GA summary

- GAs are general purpose optimisation algorithms working, in overview, as shown in Figure 16.21.
- They are based loosely on certain concepts of biological evolution (genes, chromosomes, mutations, generations, [un]natural selection and survival of the most fit for purpose).

16.4.4.1 The main steps in classical GA

- Encode the problem so that the solution sought is in the form of a binary string 0110110010... called a chromosome.
- Generate a totally random set of (say 100) chromosomes of the right length to be a solution.
- Evaluate the fitness (a single positive number) of each chromosome.

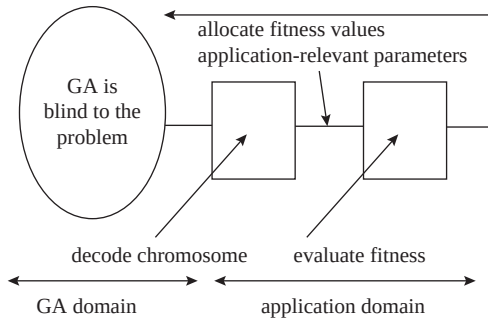


Figure 16.21 How the GA is linked to the problem through the fitness function

- Probabilistically, select the most-fit chromosomes to be parents for the next generation and produce a new generation from these parents by the crossover mechanism.
- Continue the cycle of generations until a satisfactory solution has been obtained as measured by the fitness value.

16.4.4.2 GA advantages

- The entire space is searched in parallel, avoiding the solution terminating in local minima.
- GAs are less prone to noise problems than methods that need to evaluate derivatives.
- No knowledge of the problem is needed except for calculation of fitness values.

16.4.4.3 GA disadvantages

- GA is just a general idea and many difficult application-dependent tasks have to be undertaken (particularly encoding and definition of fitness function, etc.).
- For all but demonstration problems, the computer power/time required to produce a realistic solution may be considerable.
- Much of the GA practitioner's art and skill lies in getting an algorithm to converge when faced with a large problem. Such strategies as incremental evolution (in which coarse approximations are successively produced and then refined) are subjects of current research (Kalganova, 2000).
- GAs have a poor reputation for handling constraints.

16.4.5 References

Banzhaf W., Nordin P., Keller R.E., Francone F.D. (eds.). *Genetic Programming: An Introduction* (The Morgan Kaufmann Series in Artificial Intelligence) (San Francisco, Morgan Kaufmann; Heidelburg, Dpunht Verlag, 1999)

Kalganova T. 'Bidirectional incremental evolution in evolvable hardware'. *Proceedings of the Second NASA/DoD Workshop on Evolvable Hardware*; Palo Alto, California (Piscataway, New Jersey, IEEE Computer Society, 2000)

Koza J.R., Keane M.A., Yu J., Mydlowec W., Bennett F.H. III. 'Automatic synthesis of both the topology and parameters for a robust controller for a non-minimal phase plant and a three-lag plant by means of genetic programming'. *Proceedings of IEEE Conference on Decision and Control*; Chicago, IL, 1999, pp. 5292–300

Reeves C.R. *Genetic Algorithms: A Guide to GA Theory* (Dordrecht, Kluwer, 2002)

16.4.6 Rivals to GAs? Autonomous agents and swarms

People are moved in a large city by a mixture of methods ranging from centrally planned underground trains running at scheduled times on fixed routes to a shifting mass of taxis operating largely autonomously. Agents and swarms have some similarity to these taxis: having been set going, they may together solve a very complex problem by a mixture of rivalry and co-operation.

Some specimen references are as follows:

Bonabeau E., Theraulaz G., Dorigo M. *Swarm Intelligence: From Natural to Artificial Systems* (New York, Santa Fe Institute of Studies on the Sciences of Complexity, Oxford University Press, 1999)

Ferber J. *Multi-Agent Systems: An Introduction to Distributed Artificial Intelligence.* Harlow, Addison-Wesley, 1999

16.5 Controller switching, gain scheduling and adaptivity

16.5.1 Discussion

Many real-world processes that need to be controlled, change their characteristics widely when their operational environment changes. For instance, a military fighter aircraft will surely be unable to be represented by the same constant mathematical model at take-off and at supersonic speeds and it is therefore unlikely that robust control design techniques can produce one constant control system that guarantees sufficient performance with stability over such a wide range of operating circumstances.

A robust controller is designed to control all processes having transfer functions G, in the range $G + \Delta G$, where ΔG represents either modelling uncertainty or process variability.

Where ΔG is large, the performance with any fixed robust controller may be inadequate for the application and one or other of the controller adapting or learning techniques described below may need to be used.

16.5.2 Controller switching

The simplest approach to the control of a process whose characteristics change over too large a range for any single constant controller to be effective would possibly be to store a number of pre-configured controllers with appropriate pre-configured

parameters, including gains. Not so simple is the task of deciding the mechanism that would reliably initiate the switch-over of controllers/strategies. Further, the dynamic disturbances possibly caused by the switch-overs and the spectre of continuous chattering from one controller to another exists, since there is, so far, no accepted theoretical framework that can guarantee how a multi-model/multi-controller system will perform overall.

16.5.3 Gain scheduling

Gain scheduling is the name used for a range of techniques in which the gain and other parameters in a usually closed loop controller are modified during operation according to a set of fixed stored rules. This means that a gain-scheduled controller is pre-programmed to change its parameters according to (say) airspeed or altitude for an aircraft.

Where a process model is known but is non-linear, such as the model $x = f(a, x, u)$, where a is a vector of process parameters, then one approach would be to linearise the model at $a{\cdot}n$ key points in the operational range of the process and to design, store and interpolate continuously between those controllers according to the actual operating point. More sophisticated scheduling approaches can be envisaged but they will often be infeasible in practice because of model uncertainties.

16.5.4 Adaptivity and self-tuning controllers

Clearly, adaptive controllers, at least in theory, offer a much more theoretically sound approach to meeting the challenges of maintaining high performance with sufficient stability margins over widely different sets of conditions. In laboratory conditions where an occasional mishap is not a disaster, adaptive systems can perform well. The problem for real-world applications is allowing an adaptive controller sufficient range of action and autonomy to be effective. For many safety-critical applications, there is always a fear that a rogue signal might push the adaptive controller into a danger area. Self-tuning controllers that are capable of reliable self-commissioning when connected to any unknown industrial process and which then obligingly and again reliably change their set-up parameters to track the unmonitored time variation of the unknown process are to be found on the wish lists of all process managers, but the operative word is *reliable* and most reported successes will, on investigation, be found to be simulations.

16.6 Learning systems (systems that learn) with or without supervision

16.6.1 Basic ideas

A machine that can learn by trial and error and that can refine its behaviour over time has very obvious attractions. Further, one could reasonably expect that the ever-increasing availability of increased computer power, speed and memory could enable such technologies to be developed and put into application.

Learning in its general sense involves

(1) a learner;
(2) something to be learned;
(3) examples or selections from what is to be learned displayed to the learner;
(4) trial solutions or hypothesis provided by the learner;
(5) (possibly) a teacher or a cost function to give feedback to the learner.

16.6.2 Learning versus adaptivity

Adaptivity implies that in response to a change in (say) environment, a system will modify its behaviour always in the same way no matter how many times the operation is performed. However, a learning system, in contrast, faced with a task similar to one encountered previously, can be expected to respond with increasing efficiency – at least until some asymptotic limit to learning has been reached.

16.6.3 Structural characteristics of an abstract learning system

(a) An initially empty knowledge space that will be populated by knowledge functions that have been accumulated from earlier recorded experiences. The knowledge space with its current set of knowledge functions will be called the knowledge base.
(b) A knowledge interpreter and interpolator whose aim is to build the best possible knowledge base with the minimum of experimentation. It is the task of this device to choose control strategies that when implemented will produce data rich in information to help fill the knowledge base appropriately.
(c) An objective function that defines the purpose of the whole exercise.

Figure 16.22 indicates the concept. In practice, the learning involved in finding a good control system for a new 'unknown' process, such as that occurring in the manufacture of a new pharmaceutical product, requires a large number of interacting decisions to be made as shown in Figure 16.23.

Although the procedures used will in broad principle follow the outline shown in Figure 16.22, many of the decisions to be made rely on the inherited wisdom of experts and the sequence of events is still heavily supervised by human experts as shown diagrammatically in Figure 16.24 with the emphasis being initially on finding a process model and then changing to a concentration on performance optimisation (Figure 16.25).

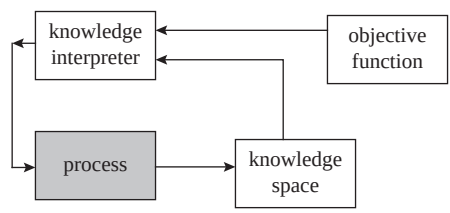


Figure 16.22 Learning control concepts – the structure of an abstract system

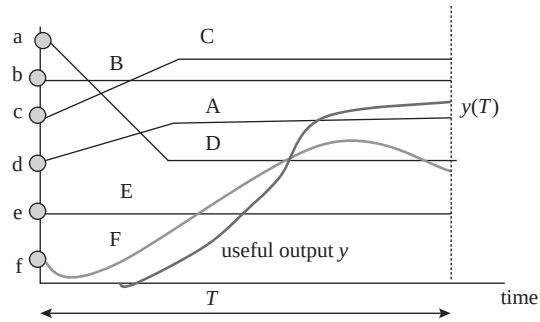
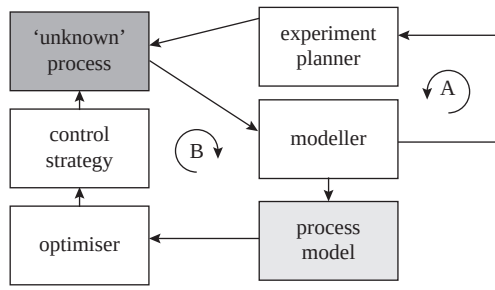


Figure 16.23 *The control strategy for a bioprocess must fix initial conditions for physicochemical variables a–e and for bioprocess variable f and stipulate trajectories A–F to be followed during the batch by these variables. The objective is to maximise yield of product y taking into account batch time T*



A: model development loop, B: performance optimisation loop

Figure 16.24 *Rapid control development for a new batch process*

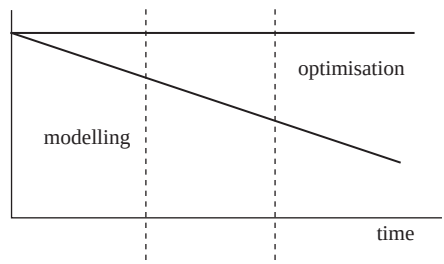


Figure 16.25 *Rapid control development for a new batch process: expected progression. Initially most time is spent gathering data and attempting to find a model that describes the process. As model development proceeds, so does control algorithm development*

It is perhaps worth pointing out that as the above procedure progresses, the physical size of the experimental reactors must also progress from laboratory scale through ever larger reactors towards near-production-scale reactors where realistic control regimes can eventually be finalised. Scaling-up studies can require care (see interlude 6H for one example).

16.7 Intelligent systems

16.7.1 The properties that an intelligent system ought to possess

The qualities and properties that an intelligent system ought to possess (based on presentations at recent IFAC (International Federation of Automatic Control) meetings) are as follows. An intelligent control system, in its most ambitious form, should possess autonomy in terms of

- self-learning
- self-reconfigurability
- reasoning under uncertainty
- planning
- decision making
- failure detection
- setting control goals (not only attaining them)

It is clear that some current systems do indeed possess several of the quoted properties.

However, it has to be admitted that the state of development of intelligent control systems as measured against the list is still quite modest – perhaps not surprisingly, given the ambition built into the list.

More ambitious still is the definition 'systems that can deliberate about the past and generate plans and strategies for the future'. Measured against this definition, achievements so far appear pedestrian indeed. However, such a scenario-generating architecture to meet that requirement has been proposed (by Albus (1997)) along the lines of Figure 16.26.

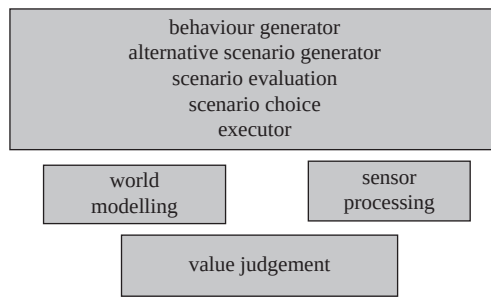


Figure 16.26 NIST-RCS (National Institute of Standards and Technology – Real-Time Control System): an architecture for intelligent system design (Albus, 1997)

If and when a computer architecture such as the one shown becomes generally available, we shall have an ideal platform to help us to rapid, reliable and transparent implementation of a wide range of intelligent control systems.

16.8 Where next for AI techniques?

Where next for the AI-inspired technologies that arrive regularly on the research and development scene?

It can be safely said that few of the new ideas that are announced will fulfil their early promise but some will make it through to application and will probably alter our lives.

At the time of writing, scans of papers being presented at current control meetings include quite significant numbers in the following linked areas: **agent-based systems, autonomous agents, swarm intelligence, co-operative control, multi-vehicle systems, networked control systems, swarm intelligence, unmanned aerial or submersible vehicles.** (Because these topics are in general a long way from the control mainstream, few, if any, references have been quoted in this book for those areas.)

It is clear that Nature is being imitated and that groups of unmanned co-operating vehicles, making clumsy attempts to behave like flocks of birds, will shortly, if not already, be able to do the nasty jobs, such as fighting wars, that civilisation still has not learned how to avoid.

Some of the applications for the technologies listed above are a long way indeed from the servomechanisms where control began. For example, there are published papers with subtitles such as 'Trust and Reputation *M*odels for Agent-based Virtual Organisations'.

16A The idea of a probing controller

Akesson and Hagander (2000) have proposed a so-called probing controller that uses a generic idea for tracking just below invisible varying and unknown constraints that occur in a batch process. The idea is to make probing pulses in the glucose feed rate and to monitor the responses that change as the constraint is approached. By this method, it is possible to detect and avoid a characteristic saturation linked to undesirable by-product formation. Figure 16.27 shows how in *Escherichia coli* fermentations, the optimal carbon feed rate will run along invisible constraints. The probing controller finds these boundaries by pulsing the feed rate as shown in Figure 16.28 and observing the nature of the response.

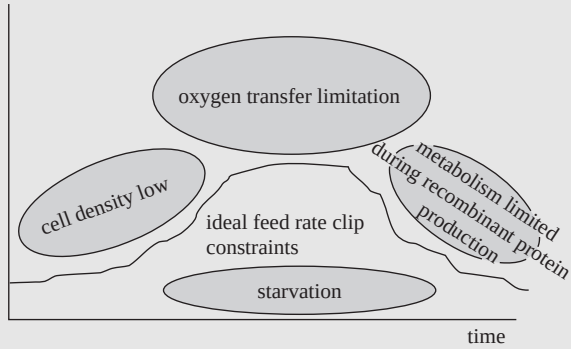


Figure 16.27 Carbon feed rate constraints in E. coli–based expression systems

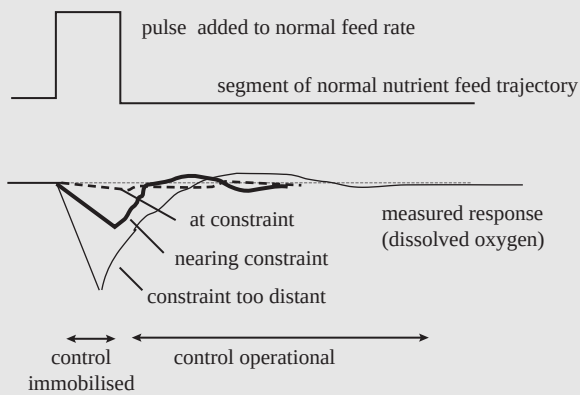


Figure 16.28 How the nearness to the constraint can be inferred from the measured responses to the injected pulses

The idea could be adapted to other processes where variable invisible constraints have to be approached as closely as possible.

Akesson M., Hagander P. 'A simplified probing controller for glucose feeding in Escherichia coli cultivations'. *Proceedings of the IEEE Conference on Decision and Control*. 2000;**5**:4520–25

Chapter 17

Review: the development of the control systems discipline and the mathematical roots of control systems theory

17.1 A rapid review of how control technology developed

During the period of early industrial development, control was not identified as anything significant since the main preoccupations were with wider basic issues. For instance, the main problems in the early coal industry were with explosions, roof falls, carbon monoxide poisoning and dust-borne diseases. Once these problems had been largely solved, control systems technology came into play, for instance, in the design of remotely operated coal cutters. Present-day coal mine managers are now preoccupied with logistics, reliability, information and maintenance. The evolutionary pattern – mechanisation/automation and control/organisation and logistics – can be discerned in almost every industry (Figure 17.1).

Thus, automatic control was scarcely needed until mechanisation had produced the devices and processes that needed to be controlled, and, in fact, it was the requirements of telephony that drove Nyquist (1932), Black (1934), Bode (1945) and co-workers to develop their frequency response and feedback techniques that were to have such wide applicability much later.

However, an early prophet of things to come wrote, 'In this age characterised by huge resources of mechanical and electrical power, these agencies have in many fields almost completely replaced human muscular power. In a similar way the functions of human operators are being taken over by mechanisms that

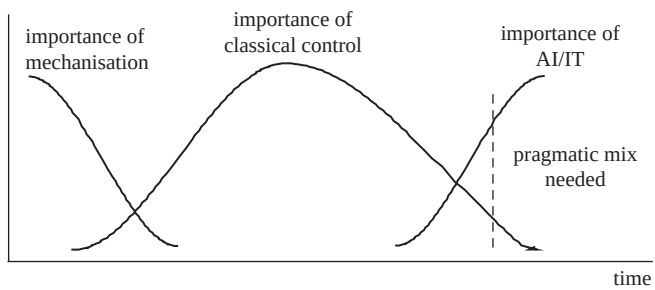


Figure 17.1 The typical evolution: mechanisation/automation/organisation

automatically control the performance of machines and processes'. So wrote H.L. Hazen in a far-sighted paper in 1934. Many of the concepts that Hazen and his contemporaries realised to be possible were slow to materialise because of the absence of reliable devices for computation and information transmission. It required the technical stimulus of World War II, and a long period of development before Hazen's ideas began to be applied in depth to the more advanced end of the industrial and business spectrum in the 1960s and 1970s. The slow growth was due to the high cost, unreliability and difficulties of application of early computers.

Since all usable systems have to be stable, stability theory is involved implicitly or explicitly in every control application and arguably this is the strongest thread that needs to extend to fully underpin the newer areas where IT, computing and control theory overlap to unify the wider control topic. Early designers of mechanical devices had to ensure stable operation through ingenious mechanical means rather then using control design approaches, which had not yet been invented. For instance, James Watt designed his governor for steam engines in 1788 (Figure 17.2). It uses a form of feedback via a velocity-dependent linkage. In practice, the Watt governors often gave poor speed control and allowed oscillatory behaviour. Maxwell (1868) derived the differential equations describing the governed system, linearised the equations about an equilibrium point and showed that the system would be stable if the roots of the characteristic equation all had negative real parts. He then converted his conclusions into recommendations to add viscous friction to damp the governors. These early examples already illustrate the still continuing trend whereby intelligence is transferred from a designer's head into a mechanism, a controller or a data base to give increased machine autonomy (Figure 17.3).

During World War II and after, new designs of aircraft, guns and missiles needed new types of control systems that stretched existing knowledge resulting in new research and new powerful techniques.

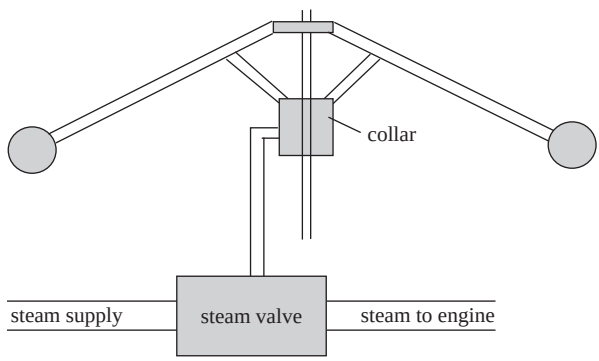


Figure 17.2 James Watt's centrifugal governor of 1788 (when the collar lifts, the valve reduces the supply of steam to the engine-feedback control)

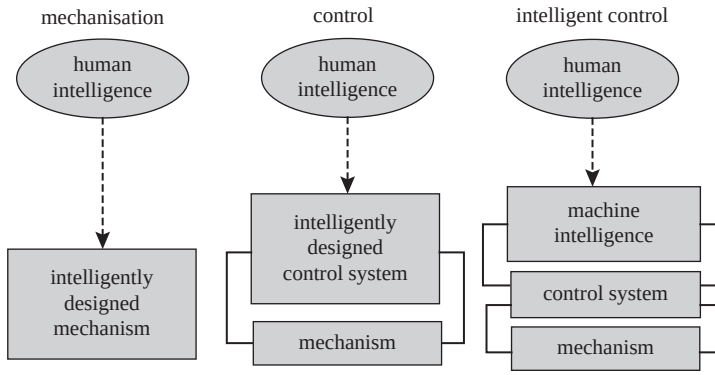


Figure 17.3 Phases of development

In the period 1945–1965, these so-called classical techniques, with heavy emphasis on graphical visualisation in the frequency domain and with mathematical underpinning by the theory of functions of a complex variable, were applied with spectacular success to industrial problems, particularly those in the oil, gas, metals and chemical industries. Most of the algorithms passed without difficulty into the computer age as discrete time versions where they still keep most of the wheels of industry turning or stationary, as required.

In the period 1960–1990, matrix-based multivariable theory, with its theoretical foundation being linear algebra and operator theory, developed in earnest and there resulted the beautiful coherent core of linear control theory. It is that coherence that guarantees the availability of transformations between different representations and domains so that, for instance, the structure, transient and frequency responses and stability characteristics of any given system can be looked at and manipulated in whichever domain is most convenient.

However, the mathematical attractiveness of control theory did not guarantee its universal commercial success.

The drivers for the development of control theory had come from the predominantly academic developers themselves with little pull from the industrial managers whose applications stood to benefit. Not surprisingly, the result was a lot of theory looking for applications and a certain amount of resulting disillusionment all round.

Quite a few problems were caused by naïve assumptions, such as the following:

- Accurate unchanging mathematical models of complex industrial processes could be produced at a non-exorbitant cost and that ‘clean’ mathematics could encode the messy realities of the world of work.
- The often ill-defined economic aims of a complex plant could reasonably be expressed as a single scalar-valued cost function, thus allowing meaningful optimisation studies to take place.

The failure in the real world of many of the highly rigorous mathematical optimisation techniques resulted in two parallel developments:

(1) The development of robust control methods, still mathematically deep, but now attempting to quantify and take into account some of the uncertainties and modelling errors that had caused the failures above.
(2) A move towards anthropomorphism with a realisation that imitating nature might have a lot to offer. This theme (imitating nature) combined with the ready availability of computing power and data-collection techniques has resulted in the appearance of a disparate set of so-called artificial intelligence (AI) techniques, some of which have been reviewed in Chapter 16.

17.2 The development of the control systems discipline: a structure

Figure 17.4(a) appears already in the introductory chapter and is reproduced here for continuity.

Figure 17.4(b) is a somewhat enhanced version and shows some additional continuity and linking of minds, in particular the following:

The enhanced figure shows that Maxwell's work on stability was triggered first, not by steam engines, but by the subject of the Adams Prize competition, set each year (at that time) by St John's College, Cambridge. In 1856, the topic was to conjecture on the nature of Saturn's rings. He solved the problem brilliantly by showing that, of the three possible 'natures' (solid, liquid, particulate), the first two would be unstable and must be dismissed as possibilities. In 1877, the Adams Prize committee, with Maxwell now a committee member, set the topic, 'The Stability of Motion'. The winner was E.J. Routh.

Figure 17.4(b) shows additional developers of the communication-related frequency response area and we suggest that it was actually the lesser-known Black who recognised the great importance of feedback, while Bode, in his 1945 paper, laid out the rigorous quantitative relationships that constrain what can and what cannot be achieved in designs in the frequency domain. Interestingly, those 'laws' of Bode were, for a time, rather buried, while multivariable systems were studied intensively in the time domain. Now that realistic robust control design, particularly using loop shaping, is in welcome use; Bode's laws supply the frequency domain boundaries limiting robust high-performance system design.

Lyapunov's 1892 doctoral thesis (University of Moscow), 'The general problem of the stability of motion', only became generally available in English when Dr Tom Fuller translated the original on the anniversary date in 1992. The importance of the Lyapunov inequality viewpoint has continued to increase year by year since then. Now, with the great strides taken by linear matrix inequality (LMI) techniques, A.M. Lyapunov's work is being seen as the initial source of a whole new branch of realistically applicable yet rigorous optimisation algorithms.

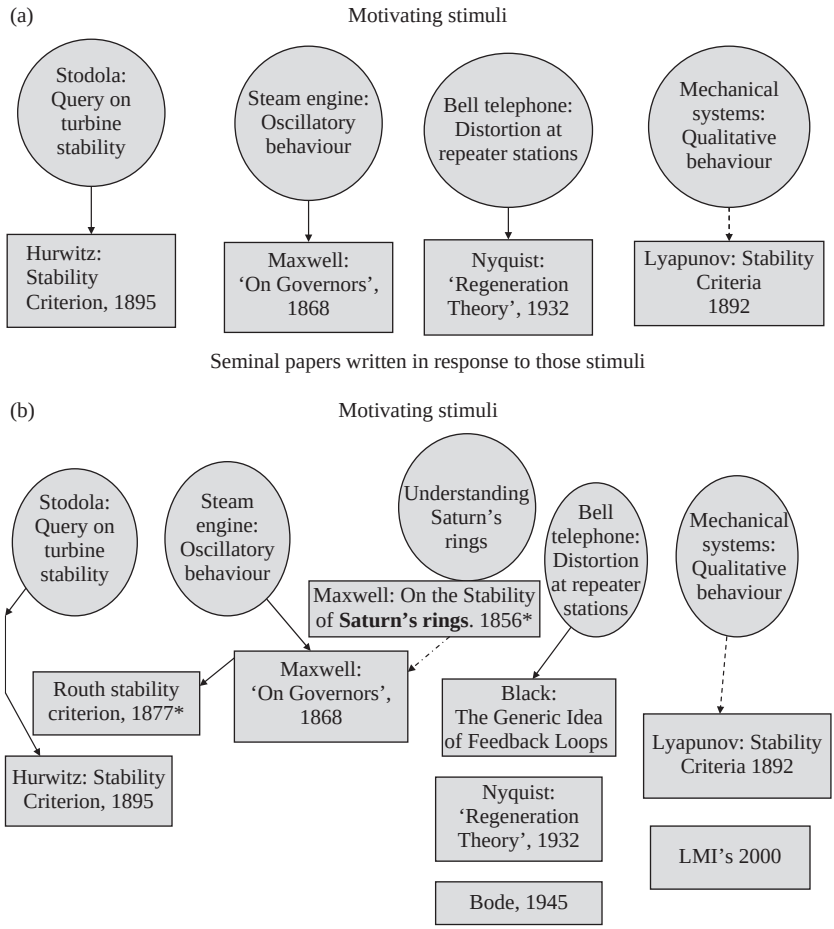


Figure 17.4 (a) Some of the significant ideas that helped to provide a structure for the control systems discipline; (b) An enhanced version of figure (a), showing further important 'linking of minds and ideas'

17.3 The mathematical roots of control theory

Control theory deals with objects called systems that are abstract, idealised generalisations of reality; this characteristic explains both the power and the intellectual appeal of the subject. Control theory rests on a very substantial mathematical foundation that gives rigour to mathematical representations of dynamical systems and underpins all the approaches to systems analysis and multivariable controller design (Table 17.1).

Three main threads of transferable development have been particularly important in supporting the development of control theory.

(i) The understanding of dynamical systems that was stimulated by inquiry into the operation and stability of the planetary system.
(ii) Developments such as function theory, complex variable theory, calculus of variations and optimisation.
(iii) Rigorous definitions of linear spaces such as Banach and Hilbert spaces, allowing theoretically sound representation and manipulation of multivariable systems.

Table 17.1 Some historical mathematical developments that helped to establish the enabling infrastructure for the present-day control systems discipline

Copernicus	1478–1543	Set the sun at the centre of the planets
Cardan	1501–1576	His 'Ars magna' uses 'imaginaries' as well as negative numbers
Kepler	1571–1630	Laws of planetary motion
Newton	1642–1727	Established celestial mechanics on a mathematical basis
Johann Bernoulli	1654–1705	His work on the brachistochrone problem (1696) has been put forward as the beginning of the calculus of variations
Daniel Bernoulli	1700–1782	His analysis of a vibrating string by superimposed overtones was the precursor of Fourier theories before Fourier was born
Euler	1707–1783	His 886 published works made an enormous contribution across a wide range of analytic mathematics
Lagrange	1736–1813	Analytic mechanics
Laplace	1749–1827	Celestial mechanics and stability
Fourier	1768–1830	'The analytical theory of heat'. Representation of arbitrary periodic functions using superimposed basis functions.
Cauchy	1789–1857	Founded the subject, functions of a complex variable
Poincaré	1854–1912	May be said to have founded the subject of topology
Hilbert	1862–1943	
Lebesgue	1875–1942	Hilbert through rigorous axiomatics and Lebesgue by his introduction of the concept of measure laid strong foundations for the families of linear spaces that underlie all state variable representations
Maxwell	1831–1879	1857 Adams Prize for his essay on the nature of Saturn's rings. Stability of spinning tops; paper 'On governors'
Lyapunov	1857–1918	1892 'General problem of the stability of motion'
Birkhoff	1884–1944	Dynamical systems and ergodic theory
Wiener	1894–1964	Founded the modern discipline of cybernetics
Pontryagin	1908–1988	1961 *Mathematical Theory of Optimal Processes* (The maximum principle)
Shannon	1916–2001	Shannon-Hartley theorem (often referred to as Shannon's law) establishing fundamental quantitative bounds for the capacity of communication channels
Kalman	1960–	Linear filtering and prediction. Major advances in state-space representations and properties of systems

Developments of specifically control-oriented techniques are not included here.

Chapter 18

Resources, references and further reading

18.1 General remarks on the control literature and on the following references and recommended further reading

Almost all the concepts and topics that together make up our subject of control systems began life as tentative ideas. Those ideas that survived the highly competitive evolutionary process by having sufficient theoretical or practical promise gradually matured and merged into the control systems mainstream.

The literature describing these ideas, topics and sub-topics follows much the same life cycle.

(1) First, a sequence of conference or journal papers appears, establishing the scientific basis for an idea and perhaps giving preliminary results from trial simulations.
(2) Next, pioneering monographs and text books will appear, often authored by the idea's originators and close contemporaries.
(3) The final stage is when the idea becomes a recognised topic, now fully developed, taking its place in the several 800-plus page all-embracing college text books. Those useful books cover basic topics very thoroughly but necessarily give a rather broad-brush coverage to the many individual topics that are not quite in the mainstream.

In providing references and further reading recommendations for this new edition I have provided a large number of references from category 2 above, since those papers and books have the freshness and spontaneity of the creative years in which they were written. Additionally, many of those references allow the interested reader to view my sources.

Where appropriate, I have provided references to historic literature that I believe provides a strong long-term intellectual foundation for our subject. Additionally, I have provided supporting links to easily accessible comprehensive college texts for the convenience of readers who are not particularly interested in the historical roots of the subject.

Similar remarks apply to the literature describing the mathematical topics that underpin the Control Systems subject. Therefore, in addition to referencing the most recent mathematic texts, I have also referenced many wonderful old texts that have often been my own sources: many of these are timeless and well worth consulting.

18.2 Finding books, papers, theses

Google Books, Google Scholar and Amazon will usually produce an abundance of leads, although free online access to contents is becoming increasingly restricted.

The control systems community is well served by regular conferences, notably the International Federation of Automatic Control (IFAC) Triennial Congresses whose archived papers are free to view online, the annual American Control Conferences and the biennial European Control Conferences. Many of the papers presented at such meetings are too specialist to be of much interest to anyone not working in the same field, but the overall programmes give a good indication of research trends.

UK theses can be found using the EThOS (Electronic Thesis Online Service).

18.2.1 Library books

The control literature is concentrated, as far as the Dewey system is concerned, into the 629.8 category with particular entries being

629.8312 Control theory
629.836 Nonlinear and adaptive control
629.895 Computer control

Other entries of interest are

511.8 Mathematical models
515.352 Ordinary differential equations
515.353 Partial differential equations
515.625 Difference equations
515.64 Calculus of variations
515.7 Functional analysis

18.3 Control-oriented software

Matlab/Mathworks product literature available on the Web is often remarkably informative and concise, even for someone who only wants to learn quickly about (say) algorithms for model reduction without necessarily wanting to carry out any computation.

Scilab and Xcos (both freeware) can be partial substitutes for Matlab and Simulink when those are not available. Scilab in particular has a huge range of capabilities but its documentation cannot match that of Matlab.

Software for symbolic solution is available from Mathematica and Maple and for some situations from Matlab.

Simulation software that is specifically designed to include hardware in the loop for prototyping is available from Vissim.

(The above are only examples and there are many other possibilities.)

18.4 Mainstream control literature

A very readable book at introductory level with many imaginative features and a host of not so simple exercises is Åström and Murray (2008).

Mainstream control is now a mature topic and this is reflected in the literature for undergraduate courses that is dominated by a few impressively large text books. Typically, these books (see below) are now (2012) in at least their fifth editions and each covers a wide range of topics including, usually, introductory material on modelling, optimisation and state estimation, often with Matlab exercises.

D'Azzo J.J., Houpis C.H., Sheldon S.N. *Linear Control System Analysis and Design with Matlab.* 5th edn (Basel, NY, Marcel Dekker, 2003) (832 pages; began life in 1960 with 580 pages)

Dorf R.C., Bishop R.H. *Modern Control Systems: International Version*.12th edn (London, Pearson Education, 2010) (1104 pages; began life in 1967 with 400 pages)

Franklin G.F., Powell J.D., Emami-Naeini A. *Feedback Control of Dynamic Systems: Plus MATLAB & Simulink Student Version*. 6th edn (London, Pearson Education, 2011) (about 850 pages; began life in the 1980s with around 600 pages)

Ogata K. *Modern Control Engineering: Plus MATLAB & Simulink Student Version*. 5th edn (London, Pearson Education, 2011) (912 pages; began life in 1967 with 600 pages)

Kuo B.C., Golnaraghi F. *Automatic Control Systems*. 9th edn (Hoboken, NJ, Wiley, 2009) (800 pages; began life in 1962 with about 450 pages)

18.5 Older mainstream control books

Many older books have a great deal to offer, having been written during the heady days (one might say, 'Golden Years') when the subject areas they describe were being created. Among the books that I have enjoyed working from are the following:

Chestnut and Mayer (1959), Horowitz (1963), Newton *et al.* (1957), Truxal (1955), Thaler and Brown (1953), Tou (1964), Zadeh and Desoer (1963) and from a little later Brockett (1970) and Wonham (1985).

Zadeh and Desoer is an indispensable book for anyone interested in a rigorous approach to control theory. Brockett is a superb book giving a simple yet advanced geometric view of systems behaviour. Wonham also gives a welcome geometric viewpoint.

More older books have been listed in the references for the reasons that they are still entirely relevant and that their coverage, approach and level of detail cannot readily be found in current books.

18.6 Methodologies for economic justification of investment in automation

Please refer to Section 9.6 for recommendations.

18.7 State estimation

Two seminal papers, Kalman (1960) and Kalman and Bucy (1963), laid firm foundations for everything that has followed since. Grewal (2008) is something of a standard text on Kalman filtering, and Chui (2009) is more introductory. See also Simon (2006).

18.8 Sliding mode control

In ***sliding mode control***, a system is designed so as to follow one or other switching surfaces, potentially yielding consistent operation despite varying application conditions. *Utkin (1977) was a pioneering author in this area, and Zak (2003) has some very interesting explanatory illustrations (some are quoted in Section 9.2.2 of this book).*

See also Bartolini (2008), Milhoub (2009), Misawa (2012), Perruquetti and Barbot (2002), Edwards and Spurgeon (1998).

18.9 Optimisation

The literature on optimisation is very extensive. The bibliography lists two books that are concerned with inequalities, since a study of these is a pre-requisite for understanding certain approaches to optimisation. The references are Beckenbach and Bellman (1961) and Hardy *et al.* (1967).

Athans and Falb (2006) and Anderson and Moore (2007) are the main recommendations for this topic.

There are useful early books on specific topics in optimisation; for instance, Hestenes (1966) on the calculus of variations, Pontryagin *et al.* (1964) on the maximum principle and Bellman (1957) on dynamic programming.

Classic texts are Bryson (2002), Markus and Lee (1967) and Sage and White (1977). Grimble and Johnson (1988) is a very comprehensive two-volume set. I have not discovered more recent straightforward books devoted to optimisation.

Finally, I mention Pallu de la Barriere (1967), which is still in print. This book, by making mathematical demands on the reader, may act as a motivator for those who need a concrete reason for studying further mathematics.

18.10 Robust control

There is a very extensive literature. A first stop to help understanding of some of the concepts could be Kwakernaak (1993) and Morari and Zafiriou (1989), while two papers by Zames (1976, 1981) show the origins of the theoretical developments.

Other useful references are Chen (2000A), Chen (2000B), Dullerud and Paganini (2000) and Zhou *et al.* (1996).

Duren (2000) covers the mathematical foundations, whereas Turner and Bates (2007) describe mathematical methods. Grimble (2001) is good on real industrial applications. Vinnicombe (2000) describes application of the ν gap metric of which he was the originator.

Chen (2011) describes a practical application; Hsu *et al.* (2006) is concerned with structured identification; Li (2004) is concerned with stochastic optimisation, Rao and Sen (2000) describes a linear matrix inequality (LMI) approach. Matlab documentation (2011) describes a case history in which a μ approach is used to design a suspension system.

Also relevant is the technique of ***quantitative feedback theory (QFT)*** pioneered by Horowitz (1993). QFT is a frequency response technique that uses feedback to compensate the effects of unmeasurable process uncertainties or non-linearities. See also Yaniv (1999).

Closely associated with robust control are the topics of *Sensitivity Analysis*, Saltelli *et al.* (2000), and *Algorithm Fragility*, Istepanian and Whidborne (2001).

18.11 Neural networks and support vector methods

A large general introduction to neural networks is Sivanandam (2006), but the intensive publication period was some years earlier, with examples listed below.

On neural networks, some theoretical background can be found in Kecman (2001), Vidyasagar (2002). The application of neural networks in dynamic modelling, estimation and control is treated in Hovakimyan *et al.* (2000).

Considerable claims have been made for alternatives to neural networks in the form of ***support vector methods, kernel methods and adaptive logic networks***. See sample references Kecman (2001), Lee and Verri (2002), Cristianini and Shawe-Taylor (2000), Shaw-Taylor (2004). However, the spotlight appears now to have gone from such approaches which have largely retired into a niche among learning methods.

18.12 Fuzzy logic and fuzzy control

See Kovacic (2005), Feng (2010) and Chen and Pham (2001) for an introduction.

Zadeh is generally regarded as the inventor of the theory of fuzzy logic; see Zadeh (1969, 1972) and Bellman and Zadeh (1970).

Mamdami (1976) created the first laboratory application and Holmblad and Ostergaard (1982) pioneered the large-scale industrial application of fuzzy control.

18.13 Genetic algorithm, genetic programming and other parallel evolutionary search methods

Introductory references are Banzhaf (1999) and Reeves (2002). Koza *et al.* (1999) show how a genetic programming approach backed up by massive computer power can synthesise complex solutions for control applications. Kalganova (2000)

describes some of the computational specialism that is involved in solving realistically sized genetic algorithm problems. Hodgson *et al.* (2004) provides comparisons of some of the alternative (structured and unstructured) genetic programming approaches to the modelling of bioprocesses.

Examples of alternative approaches using ***multi-agents*** and ***swarm intelligence*** are to be found in Ferber (1999) and Bonabeau (1999), respectively.

18.14 Intelligent and learning systems

Sources of foundation theory for learning systems are Tsypkin (1971, 1973). Recent learning applications papers are Huang *et al.* (2002).

Albus is a prominent author of forward-looking papers on intelligent machines and their architectures; see Albus and Meystel (2001) and Albus (2008).

18.15 Adaptive and model-based control

Some of the most well-known model-based approaches are described in the following seminal references:

Dynamic Matrix Control (DMC), Cutler (1981)
Model Algorithmic Control (MAC), Richalet *et al.* (1977)
Internal Model Control (IMC), Garcia and Morari (1982a, 1982b)
Generalised Predictive Control (GPC), Mohtadi (1987), Tsang and Clarke (1988)
Generic Model Control (GMC), Lee (1988)
Model Inferential Control (MIC), Parrish and Brosilow (1984)
Fast Model Predictive Control (FMPC), Coales and Noton (1956)

Other references on predictive and model-based control are Camacho and Bordons (1999), Datta (1998), Forbes *et al.* (1983), Maciejowski (2001), Matausek *et al.* (2002), Mo and Billingsley (1990) and Soeterboek (1992).

18.16 Stochastic aspects of control

All real systems operate in a probabilistic environment (wind, waves, financial, political vagaries etc.), whereas a large number of systems are designed, because it is easier, to satisfy simple deterministic criteria. The extent to which systems designed against deterministic criteria will/might satisfy probabilistic criteria needs to be considered. The usual control engineering answer is to use robust control methods but a stochastic systems viewpoint is also worth considering. See reference Li (2004).

General classic references are Aoki (1967), Papoulis (2002) and Söderstrom (2002).

18.17 Some other control topics

For ***modelling and identification*** see Davidson (1988), Godfrey (1993), Sandefor (2002), Seborg *et al.* (2003) and Söderstrom and Stoica (1989). ***Control of linear time-varying systems*** is covered by Kostas and Ioannou (1993) and of ***large-scale systems*** by Koussoulas and Groumpos (1999), Lunze (1991) and Pierre and Perkins (1993).

The control of overhead cranes travelling on horizontal tracks is important in sea-container and similar logistics. When such a crane needs to move from one position to another, the application of a simple step will often cause the suspended load to swing excessively. One approach is to apply a pre-shaped input function, designed to achieve a desired response. Such approaches are designated ***input shaping techniques***, Park *et al.* (2000, 2001), Sahinkaya (2001). Of course, input shaping finds application to a range of areas outside crane control.

18.18 General mathematics references

The books quoted here are meant to supply long-term mathematics foundation material to indirectly support control theory at various levels.

One of the most useful general texts is Riley *et al.* (2006); it covers very satisfactorily most of the mathematics required in undergraduate engineering degree courses.

Rosenbrock and Storey (1970) gives a straightforward account of mathematics for control.

Hardy (1963), Binmore (1981), the old but still useful five-volume Goursat (1964) and the French series Cartan (1971), Choquet (1969), Dieudonne (1969), Godement (1969) are all recommended.

Further texts to explore are Birkhoff and Maclane (1965), Jacobson (1963), Kelley (1955), Kelley and Namioka (1963), Mostow *et al.* (1963), Protter and Murray (1975) and Halmos (1950).

Many of the books quoted above are mathematics classics.

Both Klein (1924, 1948) and Armitage and Griffiths (1969) discuss elementary mathematics from an advanced, often geometric, viewpoint.

Lin and Segel (1988) covers a wide range of interesting generic problems across the whole range of natural sciences.

18.19 Ordinary differential equations

The formulation, properties and solution of ordinary differential equations occupy a key role in system modelling and simulation. The structural and geometric properties of ordinary differential equations underlie stability theory, state space theory, controllability and optimisation and lend central support to a wide range of research topics in control theory.

Several hundred books with the title *Ordinary Differential Equations* are published each year, so there is a vast choice. Most of those in my shelves seem to be yellow Springer publications.

Mainstream texts are Braun (1993), Coddington and Levinson (1955) and Driver (1977).

Cesari (1963), Krasovskii (1963), and Willems (1970) are concerned with stability aspects.

Cartwright and Littlewood (1947) is concerned with non-linear equations.

Van der Pol (1927) is of historical interest in the theory of oscillators.

Hirsch and Smale (1974) is a superb book that is concerned with fundamental properties.

Arnold (1989) gives a quite different treatment than can be found elsewhere. His book might justifiably have been called 'Differential equations made difficult'! However, it is a very worthwhile book dealing with elementary ideas from an advanced viewpoint.

Structural aspects are covered in different ways in Andronov (1966), Bendixson (1901), Birkhoff (1927), Lefschetz (1977), Poston and Stewart (1976) and Nemitski and Stepanov (1960).

Two papers by Abd-Ali (1975a, b) use an interesting decomposition of systems using a Helmholz approach that can be regarded as an attempt to extend model analysis to nonlinear systems.

As far as ***difference equations*** are concerned, Van der Pol and Bremner (1955) is an admirable text. It is notable that this book is still frequently cited in the literature.

18.20 Differential topology/differential geometry/differential algebra

Heinz Hopf is generally considered to be the leading historic figure in the area. Hopf (1956) is a reprint of his classic lectures of some 40 years earlier. Milnor (1997), Spivak (1965) and Guillemin and Pollack (1974) are recommended. Even a glance at any of these will make any mathematically inclined person appreciate the beauty of the topic. Differential topology is a beautiful and intuitively appealing subject that is concerned with smooth mappings from non-linear manifolds onto tangent spaces. The subject would appear to be designed for the local approximation of smooth non-linear systems but the take up in that direction was rather slow for some years, although differential geometric approaches were used by, for instance, Sussman (1972) and Brockett (1978) to generalise linear systems attributes, such as controllability, to non-linear systems. In particular, some of the geometric results of Wonham (1985) for linear systems have been made applicable to non-linear problems by Isidori (1995) and Fliess and Glad (1993).

More recently, differential algebra has been applied to non-linear control problems, for instance, by Fliess (1985). A good self-contained reference to differential algebra and its application to non-linear control problems is Conte *et al.* (1999). See also Fliess and Hazewinkel (1986).

Other books that may be found useful are Berger and Gostiaux (1988), Curtis (1985) and Lang (1985).

18.21 Theory of equations

Several delightful old books on the theory of equations are Chrystal (1964), Barnard and Child (1960), Burnside and Panton (1892), Hall and Knight (1964) and Todhunter (1904). The material in these references is scarcely to be found in later texts. Another book that contains much useful material not easy to discover elsewhere is Archbold (1970).

18.22 Operator theory and functional analysis applied to linear control

Linear multivariable control models are specially labelled examples of mapping/space configurations. Thus, the natural setting for linear control theory is in one sort of linear space or another – it is only the use of limited horizons that sometimes masks this fact. Among the many attractive features that are produced by a function–analytic viewpoint is the very strong and obvious structure that is necessarily imposed on any control problem that is posed within that framework. For instance, the hierarchy of spaces (topological, linear, metric, Banach, Hilbert) constitutes a range of settings, with decreasing generality, for control problems. For instance, the last of these, a Hilbert space setting, is the natural environment for a distributed parameter optimisation problem with quadratic cost function, whereas the first, a topological setting, is so general as to be a qualitative setting for a wide class of problems.

References quoted here are in three categories:

(i) *Those that illustrate how functional analysis is applicable to control*

Here we quote Hermes and La Salle (1969), Leigh (2007), Luenberger (1969), Porter (1966), Barrett (1963) and Rubio (1971). Of these, the book by Porter affords possibly the easiest entry into the topic. There is a very large literature in the form of papers (not quoted below) with principal authors being Balakrishnan, Butkovskii, Lions, Wang, P.K.C.

(ii) *Those that deal with application of functional analysis more generally*

Here we quote Curtain (1977) and specially point out Moore (1985). This book, concerned as it is with numerical results, necessarily bridges the gap between an idea and the realisation of that idea because of its algorithmic viewpoint. Another 'bridging' reference is Green (1969), which is concerned with integral equations.

(iii) *Those that are concerned with the subject of functional analysis per se*

Books on operator theory, linear spaces and spectral theory can be considered, for our purposes, to fall into this category. Thus, there is a large literature available from which I have selected personal favourites.

These include Akhiezer and Glazman (1961), Balakrishnan (1976), Berberian (1974), Day (1962), Dunford and Schwartz (1958) and Chatelin (1983).

The standard works on linear operators are Hille and Phillips (1957), a monumental work, and Riesz and Nagy (1971).

18.23 Books of historical interest

Early references, Poincaré (1892) on celestial motion and on stability, Maxwell (1868), Hurwitz (1895), Routh (1930), are very interesting. Maxwell set questions to students on the stability of spinning tops at a date before they had any stability criteria to help them and, while still a student, successfully proved that Saturn's rings were made up of disparate fragments by a rather general stability argument. Dr Tom Fuller has extracted and edited a number of Maxwells works related to control and stability. They form a most valuable linked set of articles and include the topics cited above (Fuller, 1986). (The work of Hurwitz is discussed in Chapter 8 of this book.)

Bode (1945) and Nyquist (1932) are source references on frequency response methods.

Bellman and Kalaba (1964) contains 13 historic control papers. Basar (2000) contains 25 annotated seminal papers ending with a paper by Zames from 1981.

Other interesting references are Evans (1950, 1954) on the invention of the root locus, Jury (1958) on early work in sampled data and Kochenburger (1950) on relay control systems. Among other general references of historic interest are Hazen (1934a, b) and Oldenbourg and Sartorius (1948).

18.23.1 History of the Riccati equation

A set of references is given in interlude 13D.

18.24 Miscellany

Guest (1961) is a pre-computer-era book containing highly practicable techniques for fitting curves to time series to achieve interpolation, extrapolation and smoothing.

Guillemin (1935, 1957) are concerned with filter synthesis. These techniques have relevance to the design of systems having particular frequency domain characteristics. (These references are chosen from a wide literature on the topic to be indicative of what is available).

Kalman *et al.* (1977) is an example of a whole genre of references concerned with general systems ideas.

Shannon and Weaver (1972) is a slim book that gives an authoritative summary of information theory. The idea that the information represented by a changing situation can be quantified at different levels of approximation by Shannon's ideas is very appealing. Control would then be seen as information capture (measurement), information flow (through a channel of sufficient capacity) and information processing in a controller. However, there are few examples of the ideas having been brought conclusively to bear on a significant control problem.

The books by Arnold Sommerfeld (1950, 1952) are included because of their superb scholarly style.

In a public lecture in the mid-1980s, I showed four graphs: all having the same time axis; all having the same single curve; a rising exponential! The four vertical axes were labelled World population, Communication speed and capacity, Speed of travel, Explosive power. The forecast: inevitable instability caused by ever-increasing gains around global feedback loops involving large-scale actions rapidly reported and escalated by high-speed information links. That was the diagnosis; the solution was the hard part! Since then, we have witnessed a sequence of major financial and political crises whose main features were: (a)They were sudden (b) They had not been forecast and even afterwards they were not convincingly explained (c) In many cases, post analysis identified some very minor event that inexplicably and very rapidly seemed to trigger a major and damaging disturbance. Of course, as control engineers , we are well aware that all those rising exponential curves mentioned above indicate inherent instability and an unstable system will respond massively to a minute trigger. I often wonder why the usually very productive and inter-disciplinary inclined control fraternity does not seem to have published anything very visible in the way of analyses of these obviously feedback-loop driven global disturbances. However, Chapter 17 of Aulin (1989) titled '. . . Social revolutions and their after effects . . . ' has some interesting diagrams showing how modern civilisations have progressed over time, with the most successful managing to steer a path between the twin evils of repression and anarchy.

Aulin (1989), Brams (1983), Bunge (1959), Dyson and Havil (2009), Glansdorff and Prigogine (1971) Linderholm (1972), Segre (1984), Prigogine *et al.* (1980), Rosen (1985), Toraldo (1981), Truesdell (1984), Wigner (1960) are some examples of books that are recommended for stimulating general interest reading.

18.25 Useful tables

A few selected sets of tables that can still compete against Google look-up are as follows: Dwight (1961), contains very comprehensive integral tables; McCollum and Brown (1965), contains extensive tables of Laplace transform pairs. Prudnikov *et al.* (1992) is a very large two-volume reference of Laplace transforms and inverses. Jolley (1961) is a comprehensive table of series together with information on their summation. Burrington and May (1958) is a useful set of statistical tables.

18.26 Alphabetical list of references and suggestions for further reading

Abd-Ali A., Evans F.J. 'Structural aspects of stability in nonlinear systems'. *International Journal Control*. 1975b;**22**(4):493–516

Abd-Ali A., Fradellos G., Evans F.J. 'Structural aspects of stability in nonlinear systems'. *International Journal Control*. 1975a;**22**(4):481–91 *[The two papers describe the work of Frank Evans on Helmholtz decompositions of non-linear systems into two parts: governing stability behaviour and periodic behaviour, respectively.]*

Abzug M.J., Eugene Larrabee E. *Airplane Stability and Control: A History of the Technologies that Made Aviation Possible* (Cambridge Aerospace Series) [Paperback] (Cambridge University Press, 2002)

Affenzeller M., Winkler S., Wagner S., Beham A. *Genetic Algorithms and Genetic Programming: Modern Concepts and Practical Applications* (Boca Raton, FL, Chapman and Hall, 2009)

Aggarwal R.P., Perera K., Pinelas S. *An Introduction to Complex Analysis* (New York, Springer, 2011)

Ahrendt W.R., Taplin J. *Automatic Feedback Control* (New York, McGraw-Hill, 1951)

Akesson M., Hagander P. 'A simplified probing controller for glucose feeding in Escherichia coli cultivations'. *Proceedings of the IEEE Conference on Decision and Control*; Sydney, 2000, vol. 5, pp. 4520–5

Akhiezer N.I., Glazman I.M. *Theory of Linear Operations in Hilbert Space* (New York, Frederick Ungar Publishing Company, 1961)

Albertos P., Sala A. *Multivariable Control Systems* (New York, Springer, 2004)

Albus J.S. 'Reverse engineering the brain'. *Proceedings of the AAAI Fall Symposium*; Washington, 2008

Albus J.S., Meystel A.M. *Engineering of Mind: An Introduction to the Science of Intelligent Systems* (New York, Wiley, 2001)

Ali M., Abbas H., Werner H. 'Controller synthesis for input-output LPV models'. Decision and Control (CDC) IEEE Conference; Atlanta, GA, 15–17 Dec 2010, pp. 7694–9

Allen D. *Economic Evaluation of Projects* (London, Institute of Chemical Engineering, 1990)

Amari S.I., Ozeki T. 'Differential and algebraic geometry of multilayer perceptrons'. *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences*; Tokyo, 2001, vol. E84-A(1), pp. 31–8, ISSN: 0916-8508

Anderson B.D.O. *Optimal Control, Linear Quadratic Methods* (Englewood Cliffs, Prentice Hall, 1989)

Anderson B.D.O., Moore J.B. *Optimal Control: Linear Quadratic Methods* (NY, Dover Publications, 2007)

Anderson J.S., Brisk M.L. 'Estimating the benefits of advanced process control'. *Proceedings of IChemE Symposium Advances in Process Control*, York, 1992

Andronov A.A., Leontovich E.A., Gordon I.I., Maier A.L. *Qualitative Theory of Second Order Dynamic Systems* (New York, John Wiley, 1973)

Andronov A.A., Witt A.A., Chaikin S.C. *Theory of Oscillations* (Oxford, Pergamon Press, 1966)

Aoki M. *Optimization of Stochastic Systems* (New York, Academic Press, 1967)

Archbold J.W. *Algebra* (Bath, Pitman Publishing, 1970)

Armitage J.V., Griffiths H.B. *A Companion to Advanced Mathematics: Parts 1 and 2* (Cambridge, Cambridge University Press, 1969) [*offers some interesting insights*]

Arnold V.I. *Mathematical Methods of Classical Mechanics* (New York, Springer-Verlag, 1978)

Arnold V.I. *Ordinary Differential Equations* (Cambridge, MA, MIT Press, 1989)

Åström K., Albertos P., Blanke M., Isidori A., Schaufelberger W., Sanz R. (eds.). *Control of Complex Systems* (London, Springer, 2001)

Åström K.J., Hägglund T. *Advanced PID control: ISA - The Instrumentation, Systems, and Automation Society* (Research Triangle Park, NC, 2006).

Athans M., Falb P.L. *Optimal Control: An Introduction to the Theory and Its Applications* (NY, Dover Publications, 2006)

Aulin A. *Foundations of Mathematical System Dynamics* (Oxford, Pergamon Press, 1989) *[A remarkable book that strays wildly, but interestingly, from what might be expected from its title.]*

Baigrie B.S. *Electricity and Magnetism: A Historical Perspective* (Westport, Conn., Greenwood Press, 2007)

Balakrishnan A.V. *Applied Functional Analysis* (New York, Springer-Verlag, 1976)

Baldwin B. 'Economic justification of process improvement and automation projects'. *IEEE Annual Textile, Fiber and Film Industry Technical Conference*; Geenville S. C., 1997, pp. 1–5

Banzhaf W., Nordin P., Keller R.E., Francone F.D. (eds.). *Genetic Programming: An Introduction* (The Morgan Kaufmann series in Artifical Intelligance) (San Francisco, Morgan Kaufmann, 1999)

Barnard S., Child J.M. *Higher Algebra* (London, Macmillan, 1960)

Barnett S. *Polynomials and Linear Control Systems* (New York, Marcel Dekker, 1983)

Barrett J.F. 'The use of functionals in the analysis of non-linear physical systems'. *Journal of Electronics and Control*. 1963;**15**:567–615

Bartolini G., Fridman L., Pisano A., Usai E. (eds.). *Modern Sliding Mode Control Theory: New Perspectives and Applications*. Lecture Notes in Control and Information Sciences (Berlin, Springer, 2008)

Basar T. *Control Theory: Twenty Five Seminal Papers (1932-1981)* (New York, IEEE Press, 2008)

Bauer M., Craig I.K. 'Economic assessment of advanced process control – A survey and framework'. *Journal of Process Control*. 2008;**8**(1):2–18

Beale G. *Non-minimum-phase-zero.pdf* [online]. Available from http://bass.gmu.edu/~gbeale/ece_720/ [*Contains derivations and plots of step responses of transfer functions having additional right-half-plane zeros and additional left-half-plane poles.*]

Beckenbach E., Bellman R. *An Introduction to Inequalities* (Berlin, Random House, L.W. Singer Company, 1961)

Bell D.J. (ed.). 'Recent mathematical developments in control'. *Proceedings of a Conference*; University of Bath, Sep 1972 (London, Academic Press) *[Contains the input shaping contribution 'How to shake a piece of string straight', by Professor Patrick Parks.]*

Bellman R. *Dynamic Programming* (Princeton, NJ, Princeton University Press, 1957)

Bellman R., Kalaba R. *The Work of Lyapunov and Poincaré* (New York, Dover Publications, 1964)

Bellman R., Zadeh L.A. *Decision-Making in a Fuzzy Environment* (Springfield, VA, National Aeronautics and Space Administration, 1970)

Bendixson I. 'On curves defined by differential equations'. *Acta Mathematica*. 1901;**24**(1):1–88

Berberian S.K. *Lectures in Functional Analysis and Operator Theory* (New York, Springer-Verlag, 1974)

Bergbreiter S. *Moving from Practice to Theory: Automatic Control after World War II*. Report HIS 285S; History of Science, University of California, Berkeley, Spring 2005 *[This remarkable 40-page report documents clearly how the foundation of IFAC and its first Congress in Moscow in 1960 set the Control Discipline on the pathway to becoming ever more theoretically based. Two quotes: (In Rudy Kalman's famous 1960 paper 'On the General Theory of Control Systems' Kalman stated that he intended his paper to initiate 'study of the pure theory of control') and (in 1967, Oldenburger wrote '… IFAC has become steadily more devoted to control theory'.) Papers tend now to be judged on the basis of mathematical elegance rather than industrial or other practical need.]*

Berger M., Gostiaux B. *Differential Geometry: Manifolds, Curves and Surfaces* (Berlin, Springer-Verlag, 1988)

Beveridge G.S.G., Schechter R.S. *Optimization: Theory and Practice* (New York, McGraw-Hill, 1970)

Binmore K.G. *Topological Ideas* Book 2 (Cambridge, Cambridge University Press, 1981)

Birkhoff G., Maclane S. *A Survey of Modern Algebra* (New York, Macmillan, 1965)

Birkhoff G.D. *Dynamical Systems* (Providence, RI, AMS, 1927)

Black H.S. 'Stabilized feed-back amplifiers'. *Transactions of the American Institute of Electrical Engineers*. 1934:**53**(1):114–120 [*This paper describes and explains the theory of the Feed-back principle and demonstrates how stability of amplification, reduction of modulation products, and certain other advantages follow when stabilised feed-back is applied to an amplifier. The underlying principle of design also is set forth. The paper concludes with some examples of results obtained on amplifiers which have been built employing this new principle*.]

Blum C., Merkle D. *Swarm Intelligence: Introduction and Applications* (Berlin, Springer, 2008)

Bobal V. *Digital Self-tuning Controllers: Algorithms, Implementation and Applications*. Advanced Textbooks in Control and Signal Processing (London, Springer, 2005)

Bode H.W. *Network Analysis and Feedback Amplifier Design* (New York, Van Nostrand, 1945)

Bode H.W. 'Feedback – the history of an idea'. *Proceedings of the Symposium on Active Networks and Feedback Systems* (New York, Polytechnic Institute of Brooklyn, 1960)

Bonabeau E., Dorigo M., Theraulaz G. *Swarm Intelligence: from Natural to Artificial Systems* (Santa Fe Institute Studies on the Sciences of Complexity) (Oxford, Oxford University Press, 1999)

Bourbaki N. *Elements of Mathematics* (eight interrelated textbooks from an immortal author) (Berlin, Springer-Verlag, 1988)

Boyd S., Ghaoui L.E., Feron E., Balakrishnan V. *Linear Matrix Inequalities in System and Control Theory* (Philadelphia, Society for Industrial Mathematics, 1994)

Brams S.J. *Superior Beings – If They Existed, How Would We Know?* (New York, Springer-Verlag, 1983)

Braun M. *Differential Equations and Their Applications* (New York, Springer, 1993)

Brennan D.J. *Process Industry Economics* (London, Institute of Chemical Engineering, 1997)

Bridgman P.W. *Dimensional Analysis* New York, AMS Press, 1978 (also purchasable online at http://www.forgottenbooks.org/)

Brisk M.L. Process control: theories and profits'. *Proceedings of IFAC Congress*; Sydney, 1993, vol. 7, pp. 241–50

Bristol E.H. 'On a new measure of interaction for multivariable process control'. *IEEE Transactions on Automatic Control*. 1966;**11**(1):133-134

Brockett R.M. 'Poles, zeros and feedback-state space interpretation'. *IEEE Transactions on Automatic Control*. 1965;**AC–10**:129–35

Brockett R. W. *Finite Dimensional Linear Systems* (New York, John Wiley, 1970)

Brockett R.W. 'Feedback invariants for nonlinear systems'. *Proceedings of the VIIth IFAC World Congress*; Helsinki, 1978, pp. 1115–20

Brown G.S. 'Educating electrical engineers to exploit science', *Electrical Engineering.* 1955;**74**(2):110–115 [*On the specific task of educating feedback control engineers, Brown stated the following: 'Adequate scholarly accomplishment in the field of feed-back control requires advanced work in the fields of mathematics, physics, chemistry, measurements, communications and electronics, servo-mechanisms, energy conversion, thermodynamics and computational techniques, in addition to undergraduate training in physics or engineering. Familiarity with mathematics of differential equations, functions of a complex variable, statistics and non-linear techniques … are basic to the (learning) program'.*]

Brown G.S., Campbell D.P. *Principles of Servomechanisms* (New York, John Wiley, 1948)

Bryson A.E. *Applied Linear Optimal Control: Examples and Algorithms* (Cambridge, Cambridge University Press, 2002)

Bunge M. *Causality* (Cambridge, MA, Harvard University Press, 1959)

Burnside W.S., Panton A.W. *The Theory of Equations* (Dublin, Hodges, Figgis & Co., 1892)

Burrington R.S., May JR D.C. *Handbook of Probability and Statistics with Tables* (Ohio, Handbook Publishers, Inc., 1958)

Calltech. (2012) *Limits of control and Bode's integral formula [Robustness achieved through feedback control is subject to limits.]* A Power Point presentation available online at www.powershow.com/.../Section_4_Limits_of_Control_and_Bodes__Integral_Formula_flash_ppt_presentation

Cao Zhi-wen, Du Li-juan. 'Roll eccentricity compensation control based on FFT'. *Industrial Instrumentation & Automation*. 2006;(6):43–5

Camacho E.F., Bordons C. *Model Predictive Control*. 2nd edn (London, Springer, 2004)

Carl B. *Entropy, Compactness and the Approximation of Operators* (Cambridge, Cambridge University Press, 1990)

Cartan H. *Differential Calculus* (London, Kershaw Publishing Company Ltd., 1971)

Cartwright M.L., Littlewood J.E. 'On nonlinear differential equations of the second order'. *Annals of Mathematics*. 1947;**2**(48):472–94

Ceroni J.A. *Economic Rationalization of Automation Projects*. Springer Handbook of Automation, Part E (New York, Springer, 2009) pp. 699–713

Cesari L. *Asymptotic Behaviour and Stability Problems in Ordinary Differential Equations* (Berlin, Springer-Verlag, 1963)

Chang S.S.L. 'Statistical design theory for digital-controlled continuous systems'. *AIEE Transactions Part 2*. 1958;**77**:191–201

Chatelin F. *Spectral Approximations of Linear Operators* (New York, Academic Press, 1983)

Chen B.M. *Robust and H_∞ Control.* Communications and Control Engineering Series (London, Springer, 2000a)

Chen C.T. *Linear System Theory and Design* (New York, Holt-Saunders Japan, 1984)

Chen G., Pham T.T. *Introduction to Fuzzy Sets, Fuzzy Logic, and Fuzzy Control Systems* (Boca Ratan, CRC Press, 2001)

Chen J., Gu G. *Control-Oriented System Identification: An H_∞ Approach* (New York, John Wiley, 2000b) *[Describes methodologies for obtaining the types of models of processes and their uncertainties that are needed by H_∞ design methods.]*

Chen L., Tang C., Zhilin L. 'Study on robust control for engine idle speed based on structured singular value mu-theory'. Third International Conference on Measuring Technology and Mechatronics Automation; Shanghai, China. 2011. Available from http://doi.ieeecomputersociety.org/10.1109/ICMTMA.2011.698. ISBN: 978-0-7695-4296-6 *[The method was applied to the design on an engine idling-speed controller.]*

Chestnut H., Mayer R.W. *Servomechanism and Regulating System Design* (New York, John Wiley, 1959)

Choi Z., Johnson M.A., Grimble M.J. 'Polynomial LQG control of back-up-roll eccentricity gauge variations'. *Automatica*. 1994;**30**(6):975–92

Choquet G. *Geometry in a Modern Setting* (London, Kershaw Publishing Company Ltd, 1969)

Cristianini N., Shawe-Taylor J. *An introduction to Support Vector Machines: And Other Kernel-Based Learning Methods* (Cambridge, Cambridge University Press, 2000)

Chrystal G. *Textbook of Algebra*. Vols. 1 and 2 (New York, Chelsea Publishing Company, 1964) *[This is a modern reprint of the 1886 original. The reference (like other similar ones in this list) is included because these wonderful old mathematics books are still relevant, are scholarly as opposed to glib, and (sometimes usefully) deal with specifics rather than with generalities.]*

Chui C.K., Chen G. *Kalman Filtering: With Real-Time Applications* (New York, Springer-Verlag, 2009)

Clark R.N. 'The Routh-Hurwitz stability criterion revisited'. *IEEE Control Systems Magazine*. 1992;**12**(3):119–20

Cluett W.R. *From Plant Data to Process Control*. Kindle edn (London and New York, Taylor and Francis, 2007)

Coales J.F., Noton A.R.M. 'An on-off servomechanism with predicted changeover'. *Proceedings of IEE*. 1956;**128**:227–32 *[One of the first applications of model predictive control.]*

Coddington E.A., Levinson N. *Theory of Ordinary Differential Equations* (New York, McGraw-Hill, 1955)

Conte G., Moog C.H., Perdon A.M. *Nonlinear Control Systems* (Springer Lecture Notes in Control and Information Sciences, Vol. 242, 1999)

Cormack R.S., Mantel N. 'Doubt and certainty in statistics'. *Journal of Royal Society of Medicine.* 1990;**83**(3):136–7

Craig I.K., Henning R.G.D. 'Evaluation of advanced industrial control projects: a framework for determining economic benefits'. *Control Engineering Practice*. 2000;**8**(7):769–80

Cramer H. *Mathematical Methods of Statistics* (Princeton, NJ, Princeton University Press, 1966)

Curtain R.F., Pritchard A.J. *Functional Analysis in Modern Applied Mathematics* (London, New York, Academic Press, 1977)

Curtis W.D., Miller F.R. *Differential Manifolds and Theoretical Physics* (Orlando, FL, Academic Press, 1985)

Cutler C.R. 'Dynamic matrix control of imbalanced systems'. *ISA Trans*. 1982;**21**(1):1–6

Damm T., Homeyer J. *Systems and control theory (02/2/2010) Tutorial 7* [web downloadable]. 2010. *[An untitled example that explains nicely the controllability and the observability Gramian matrices.]*

Datta A. *Adaptive Internal Model Control* (London, Springer-Verlag, 1998)

Davidson H. 'Statistical methods for model development' in Savas E.S. (ed.). *Computer Control for Industrial Processes* (New York, McGraw-Hill, 1965)

Day M. *Normed Linear Spaces* (New York, Academic Press, 1962)

D'Azzo J.J., Houpis C.H., Sheldon S.N. *Linear Control System Analysis and Design with Matlab.* 5th edn (Basel, NY, Marcel Dekker, 2003)

Desborough L., Miller R. 'Increasing customer value of industrial control performance monitoring—Honeywell's experience'. In *Sixth International Conference on Chemical Process Control*. AIChE Symposium Series Number 326 (Vol. 98), pp. 172–192, 2002

Dietrich. 'Is there a theory of everything?' *Bulletin of the IMA*. 1994;**30**(11/12):166–70

Dieudonne J. *Linear Algebra and Geometry* (Paris, Hermann, Publishers in Arts and Science, 1969)

Dieudonne J. *Infinitesimal Calculus* (Paris, Hermann, Publishers in Arts and Science, 1971)

Doetsch K.H. *The Time Vector Method for Stability Investigations* (Royal Aircraft Establishment no. 2495, UDC no. 533.6.054, 1953)

Dorigo M., Theraulaz G. *Swarm Intelligence: From Natural to Artificial Systems* (Oxford, England, Oxford University Press, 1999)

Dorf R.C., Bishop R.H. *Modern Control Systems*. 12th edn (Boston, Mass.; London, 2011)

Driver R.D. *Ordinary and Delay Differential Equations* (New York, Springer-Verlag, 1977)

Dudley R. 'Mechanisms and implications of animal flight manoeuvrability'. *Integrative and Comparative Biology*. 2002;**42**(1):135–140

Dullerud G.E., Paganini F.G. *A Course in Robust Control Theory: A Convex Approach* (New York, Springer-Verlag, 2000)

Dunford N., Schwartz J.T. *Linear Operators, Part 1: General Theory* (New York, Inter-Science Publishers, 1957)

Dunford N., Schwartz J.T. *Linear Operators* (New York, Interscience Publishers, Vol. 1, 1958)

Duren P.L. *Theory of Hp Spaces* (New York, Dover Publications, 2000) *[A reissued Academic Press book.]*

Dwight H.B. *Tables of Integrals and Other Mathematical Data*. 4th edn (NewYork, Macmillan, 1961)

Dyson F., Havil J. *Gamma: Exploring Euler's Constant* (Princeton, NJ, Princeton University Press, 2009)

Edwards C., Spurgeon S.K. *Sliding Mode Control: Theory and Applications* (London, Taylor and Francis, 1998)

Evans W.R. 'Control systems synthesis by locus methods'. *Transactions of the AIEE*. 1950;**69**:66–9. AIEE Preprint, 50–51, January 1950 *[Pioneering work on the root-locus method.]*

Evans W.R. *Control-System Dynamics* (New York, McGraw-Hill, 1954)

Feng G. *Analysis and Synthesis of Fuzzy Control Systems: A Model Based Approach* (Boca Raton, FL, CRC Press, 2010)

Ferber J. *Multi-Agent Systems: An Introduction to Distributed Artificial Intelligence* (Harlow, Addison-Wesley, 1999)

Fieguth P. *Statistical Image Processing and Multidimensional Modeling* (New York, Springer, 2010) *[Chapter 4 contains a comprehensive derivation of the Kalman filter.]*

Fliess M. 'A new approach to the non-interacting control problem in nonlinear systems theory'. *Proceedings of the 23rd Allerton Conference*; Monticello, 1985, pp. 123–9

Fleisch D.A. *A Student's Guide To Maxwell's Equations* (Cambridge, Cambridge University Press, 2008)

Flügge-Lotz I. *Discontinuous Automatic Control* (Princeton, NJ, Princeton University Press, 1953) *[Early relay control systems using a sliding mode control system were used in German WW2 flying bombs against London.]*

Forbes J.F., Jacobson B.A., Rhodes E., Sullivan G.R. 'Model based control strategies for commercial grain drying systems'. *Canadian Journal of Chemical Engineering*. 1983;**62**(6):773–9

Fliess M., Glad S.T. *An Algebraic Approach to Linear and Nonlinear Control* (Boston, Birkhauser, 1993)

Fliess M., Hazewinkel M. (eds.). *Algebraic and Geometric Methods in Nonlinear Control Theory* (Dordrecht, Reidel, 1986)

Franklin G.F., Powell D.J., Emami-Naeni A. *Feedback Control of Dynamic Systems: Plus MATLAB & Simulink Student Version* (Pearson Education, 2011)

French A.P. *Einstein: A Centenary Volume* (London, Heinemann, 1979)

Freudenberg J., Looze D. 'Right half plane poles and zeros and design tradeoffs in feedback systems'. *IEEE Transaction on Automatic Control*. 1985;**30** (6):555–65

Fuller A.T. 'James Clerk Maxwell's manuscripts relating to control and stability'. *International Journal of Control*. 1979–86;**35**(5):729–44; **35**(5):785–805; **36**(4):547–74; **37**(6):1197–238; **39**(4):619–56; **43**(3):805–18; **43**(4):1135–68; **43** (5):1593–612

Garcia C., Morari M. 'Internal model control - 1. Unifying review and some new results'. *Journal Industrial and Engineering Chemistry, Process Design and Development*. 1982a;**21**(2):308–23

Garcia C., Morari M. 'Multivariable internal model control'. *Annual Meeting, American Institute of Chemical Engineers*, paper N 10A, Los Angeles, CA, AIChE, 1982b

Gardner M.F., Barnes J.L. *Transients in Linear Systems* (New York, John Wiley and Sons, 1942)

Gasparyan O. *Linear and Nonlinear Multivariable Feedback Control: a Classical Approach* (New York, John Wiley, 2008)

Gibson J.E. *Nonlinear Automatic Control* (New York, McGraw-Hill, 1963)

Giullemin E.A. *Synthesis of Passive Networks* (New York, John Wiley, 1957)

Glad T. Ljung L. *Control Theory: Multivariable and Nonlinear Methods* (London, NY, Taylor and Francis, 2000)

Glansdorff P., Prigogine I. *Thermodynamic Theory of Structure, Stability and Fluctuations* (London, John Wiley, 1971)

Godement R. *Algebra* (London, Kershaw Publishing Company, 1969)

Godfrey K. (ed.). *Perturbation Signals for System Identification* (New York, Prentice Hall, 1993)

Goursat E. *A Course in Mathematical Analysis* (New York, Dover Publications, 1964)

Graham D., Mcruer D. *Analysis of Nonlinear Control Systems* (New York, John Wiley, 1961)

Green C.D. *Integral Equation Methods* (London, Nelson, 1969)

Grewal M.S., Andrews A.P.*Kalman Filtering: Theory and Practice Using Matlab* (Oxford, Wiley-Blackwell, 2008)

Grimble M.J., Johnson M.A. *Optimal Control and Stochastic Estimation*. Vols. I and II (Chichester, John Wiley, 1988)

Grimble M.J. *Industrial Control Systems Design* (Chichester, John Wiley, 2001) *[A monumental piece of work based on H_2 and H_∞* methods and their application to industrial problems.]

Guest P.G. *Numerical Methods of Curve Fitting* (Cambridge, Cambridge University Press, 1961) *[Highly practical numerical algorithms are given in detail.]*

Guillemin E.A. *Communication Networks* (New York, John Wiley, 1935)

Guillemin V., Pollack A. *Differential Topology* (London, Prentice Hall, 1974)

Hairer E., Wanner G. *Analysis by Its History*. UTM Series (New York, Springer, 1996) *[A fascinating way of presenting the material.]*

Hall H.S., Knight S.R. *Higher Algebra* (London, Macmillan, 1964)

Halmos P.R. *Measure Theory* (New York, Van Nostrand, 1950)

Halmos P.R. *Introduction to Hilbert Space* (New York, Chelsea Publishing Company, 1957)

Hand D., Mannila H., Smyth P. *Principles of Data Mining* (Cambridge, MA, MIT Press, 2001)

Hardy G.H. *A Course of Pure Mathematics* (Cambridge, Cambridge University Press, 1963)

Hardy G.H., Littlewood J.E., Polya G. *Inequalities* (Cambridge, Cambridge University Press, 1967) *[Inequalities are the basis of many optimisation methods.]*

Hazen H.L. 'Theory of servomechanisms'. *Journal of the Franklin Institute*. 1934a;**218**:279–331

Hazen H.L. 'Design and test of a high-performance servomechanism'. *Journal of the Franklin Institute*. 1934b;**18**:543–80

Hawkins W.M., Fisher T.G. *Batch Control Systems: Design, Application, and Implementation.* 2nd edn, (Research Triangle Park, NC, ISA, 2006)

Hearns G., Blanke M. 'Quantitative analysis and design of a rudder roll damping controller'. *Proceedings of the 4th IFAC Symposium on Control Applications in Marine Systems (CAMS'98)*; Fukuoka, Japan, 27–30 Oct 1998, pp.115–20

Hermes H., Lasalle J.P. *Functional Analysis and Time Optimal Control* (New York, Academic Press, 1969)

Hestenes M.R. *Calculus of Variations and Optimal Control Theory* (New York, John Wiley, 1966)

Hille E., Phillips R.S. *Functional Analysis and Semi-Groups* (Providence, RI, American Mathematical Society, 1957)

Hingston P.F., Luigi C., Barone Z.M. *Design by Evolution: Advances in Evolutionary Design* (Berlin, London, Springer, 2008)

Hirsch M.W., Smale S., Devaney R.L. *Differential Equations, Dynamical Systems, and an Introduction to Chaos* (Boston, Academic Press, 2003)

Hodgson B.J., Taylor C.N., Ushio M., Leigh J.R., Kalganova T., Baganz F. 'Intelligent modelling of bioprocesses: a comparison of structured and unstructured approaches'. *Bioprocess and Biosystems Engineering*. 2004;**26**(6):353–359

Holmblad L.P., Ostergaard J.J. 'Control of a cement kiln by fuzzy logic' in Gupta M.M., Sanchez E. (eds.). *Fuzzy Information and Decision Processes* (Amsterdam, North Holland, 1982), pp. 389–99

Hopf H. *Differential Geometry in the Large*. Lecture Notes in Mathematics (Berlin, Springer-Verlag, 1956)

Horowitz I.M. *Synthesis of Feedback Systems* (New York, Academic Press, 1963)

Horowitz L.M. *Quantitative Feedback Design* (Boulder, CO, QFT Publications, 1993)

Hovakimyan N., Lee H., Calise A. 'Approximate NN realization of an unknown dynamic system from its input-output history'. *Proceedings of the American Control Conference*; Chicago, IL, 2000, vol. 2, pp. 919–23

Hsu K., Vincent T., Poolla K. 'Robust structured nonlinear system identification'. *45th IEEE Conference on Decision and Control*; San Diego, CA, 13–15 Dec 2006

Huang S.N., Tan K.K., Lee T.H. 'Iterative learning algorithm with a quadratic criterion for linear time-varying systems'. *Proceedings of the Institution of Mechanical Engineers. Part I: Journal of Systems and Control Engineering*. 2002;**216**(3):309–16

Hurewitz W. *Lectures on Ordinary Differential Equations* (Cambridge, MA, MIT Press, 1958)

Hurwitz A. 'On the conditions under which an equation has only roots with negative real parts'. *Mathematische Annalen*. 1895;**46**:273–80 *[Hurwitz' German original of the paper carries the footnote 'Herr Stodola benutzt mein Resultat... bei der Turbinenanlage des Badeortes Davos mit glänzendem Erfolge.... This translates as: 'Herr Stodola used my results at the turbine installation in the spa town of Davos with spectacular results'.]*

Isidori A. *Nonlinear Control Systems* 3rd edn (Berlin, Springer, 1995)

Isidori A. *Nonlinear Control Systems II* (London, Springer, 1999)

Isidori A., Astolfi A., Marconi L. (eds.). *Analysis and Design of Nonlinear Control Systems* (Berlin, New York, Springer, 2008)

Isidori A., Lamnabhi-Lagarrigue F., Respondek W. (eds.). *Nonlinear Control in the Year 2000* (London, Springer, 2001)

Istepanian R.S.H., Whidborne J.F. *Digital Controller Implementation and Fragility* (London, Springer-Verlag, 2001)

Jacobson N. *Lectures in Abstract Algebra* (Princeton, NJ, D. Van Nostrand, 1953)

Jai A.E., Pritchard A.J. *Sensors and Controls in the Analysis of Distributed Systems* (Chichester, Ellis Horwood, 1988)

James H.M., Nichols N.B., Phillips R.S. *Theory of Servomechanisms* (New York, McGraw-Hill, 1947)

John F. *Partial Differential Equations* (New York, Springer-Verlag, 1975)

Johnson A.M., Moradi M.H. (eds.). *PID Control: New Identification and Design Methods* (London, Springer, 2005)

Jolley L.B.W. *Summation of Series* (New York, Dover Publications, 1961)

Jones J.G. 'An equivalence between deterministic and probabilistic design criteria for linear systems'. *Journal of Sound and Vibration*. 1988;**125**(2): 341–56

Jury E.I. *Sampled-Data Control Systems* (New York, Wiley, 1958) *[Jury is credited with the invention of the Z transform.]*

Kalganova T. 'Bidirectional incremental evolution in evolvable hardware'. *Proceedings of the 2nd NASA/DoD Workshop on Evolvable Hardware*, Palo Alto, CA, IEEE Computer Society, 2000

Kalman R.E. 'A new approach to linear filtering and prediction problems'. *Journal of Basic Engineering*. 1960;**82**(Series D);35–45

Kalman R.E. 'Mathematical description of linear dynamical systems'. *Journal of the Society for Industrial and Applied Mathematics, Series A: Control.* 1963;**1:**152–92.

Kalman R.E. 'When is a linear control system optimal'. *Transactions of the ASME Journal of Basic Engineering*. 1964;**86**:51–60

Kalman R.E. 'Mathematical description of linear dynamical systems'. *Proceedings of the IFAC Congress*; London (London, Butterworth, 1966)

Kalman R.E., Arbib M., Falb P.L. *Topics in Mathematical System Theory* (NewYork, McGraw-Hill, 1969) *[Kalman probably did more than anyone to inspire and create the framework of modern control theory.]*

Kalman R.E., Bertram J.E. 'Control system analysis and design via the second method of Lyapunov'. *Transactions of the ASME Journal of Basic Engineering*. 1960;**82**:371–93

Kalman R.E., Bucy R.S. 'New results in linear filtering and prediction theory'. *Transactions of the ASME Journal of Basic Engineering*. 1961;**83**:95–108

Kecman V. *Learning and Soft Computing: Support Vector Machines, Neural Networks, and Fuzzy Logic Models* (Cambridge, MA, MIT Press, 2001)

Keel L.H., Bhattacharyya S.P. 'Controller synthesis free of analytical models: three term controllers'. *IEEE Transactions on Automatic Control*. 2008;**53** (6):1353–69

Kelley J.L. *General Topology* (Princeton, NJ, D. Van Nostrand, 1955)

Kelley J.L., Namioka I. *Linear Topological Spaces* (Princeton, NJ, D. Van Nostrand, 1963)

Klein F. *Arithmetic, Algebra, Analysis* (New York, Dover Publications, 1924)

Klein F. *Geometry* (New York, Dover Publications, 1948) *[The two reprints contain Klein's masterly exposition of 'elementary mathematics from an advanced viewpoint'.]*

Kochenburger R.J. 'A frequency response method for analyzing and synthesizing contactor servomechanisms'. *Transactions of AIEE.* 1950;**69**:270–83. AIEE Reprint, 50–51, January 1950 *[Pioneering work on the describing function approach applied to loops containing relays.]*

Kostas S.T., Ioannou P.A. *Linear Time-Varying Systems: Control and Adaptation* (Englewood Cliffs, Prentice-Hall, 1993)

Koussoulas N.T., Groumpos P.P. (eds.). *Large Scale Volumes* (Oxford, Pergamon Press, 1999)

Kovačić Z., Bogdan S. *Fuzzy Controller Design: Theory and Applications* (Boca Raton, CRC Press/Taylor & Francis, 2005)

Kowalski M.E., Jin J.-M. 'Karhunen–Loeve based model order reduction on non-linear systems'. *IEEE Antennas and Propagation Society, AP-S International Urbana (Digest)*. 2002;**2**:552–5

Koza J.R., Keane M.A., Yu J., Mydlowec W., Bennett III F.H. 'Automatic synthesis of both the topology and parameters for a robust controller for a non-minimal phase plant and a three-lag plant by means of genetic programming'. *Proceedings of 1999 IEEE Conference on Decision and Control*. pp. 5292–300 *[Koza and co-workers have been prolific in producing many other papers describing impressive feats of GA driven synthesis of filters, algorithms and metabolic pathways.]*

Kranc G.M., Sarachik P.E. 'An application of functional analysis to the optimal control problem'. *Journal of Basic Engineering*. 1963;**85**:143–50

Krasovskii N.N. *Stability of Motion* (Stanford, CA, Stanford University Press, 1963)

Kuo B.C., Golnaraghi F. *Automatic Control Systems*. 9th edn (Wiley, 2009)

Kwakernaak H. 'Robust control and H-optimisation – tutorial paper'. *Automatica.* 1993;**29**(2):255–73 *[This paper is highly recommended as a first introduction to the topic.]*

Lane P.G. 'Matching conditions for transient outputs' in Zakian V. (ed.). *Control Systems Design: A New Framework* (London, Springer, 2005)

Lang S. *Differential Manifolds* (New York, Springer-Verlag, 1985)

Lauer H., Lesnick R., Matson L.E. *Servomechanism Fundamentals* (New York, McGraw-Hill, 1947)

Laurain V., Tóth R., Gilson M., Garnier H. 'Direct identification of continuous-time linear parameter-varying input/output models'. *IETControl Theory & Applications.* 2011;**5**(7):878–88

Lee P.L., Sullivan G.R. 'Generic model control (GMC)'. *Computers and Chemical Engineering*. 1988;12(6):573–80

Lee R.C.K. *Optimal Estimation, Identification, and Control* (Cambridge, MA, MIT Press, 1964)

Lee S.W., Verri A. (eds.). 'Pattern recognition with support vector machines'. *First International Workshop, SVM 2002*; Niagara Falls, Canada. Lecture Notes in Computer Science (New York, Springer, 2002)

Lefschetz S. *Differential Equations: Geometric Theory* (New York, Dover Publications, 1977)

Leigh J.R. *Essentials of Nonlinear Control*. IEE Topics in Control Series (London, Peter Peregrinus, 1983)

Leigh J.R. *Modelling and Control of Fermentation Processes* (London, Peter Peregrinus, 1987a)

Leigh J.R. *Temperature Measurement and Control, with Consideration of Energy Conservation and Atmospheric Pollution* (London, Peter Peregrinus, 1987b)

Leigh J.R. *Artificial Intelligence and Automatic Control*. ISBN 1 4116 2354 1, 2005. Available from Lulu.com

Leigh J.R. *Applied Digital Control: Theory, Design and Implementation*. 2nd edn (New York, Dover Publications, 2006) *[Being a reprinting of the 1992 book published by Prentice-Hall International.]*

Leigh J.R. *Functional Analysis and Linear Control Theory* (New York, Dover Publications, 2007) *[Being a reprinting of the 1980 book Vol. 156, International Series, Mathematics in Science and Engineering (Academic Press)]*

Leigh J.R. *Modelling and Simulation* (London, Peter Peregrinus, 2009) *[Lulu.com reprint of the 1983 book first published in the IEE Topics in Control Series.]*

Leigh J.R., Thoma M. (eds.). 'Control in bioprocessing'. *IEE Proceedings,*. vol. 133, Pt. D, no. 5, 1986;(Special Issue)

Leonov G.A., Bragin V.O., Kuznetsov N.V. 'Algorithm for constructing counterexamples to the Kalman problem'. *Mathematics and Statistics*, *Doklady Mathematics*. 2010;**82**(1):540–2

Li T.Y., Yorke J.A. 'Period three implies chaos'. *American Mathematical Monthly*. 1975;**82**:985–92

Li X. 'Probabilistic robust control system design by stochastic optimization'. Doctoral Thesis (The Pennsylvania State University, 2004)

Libby C.C. *Motor Selection and Application* (New York, McGraw-Hill, 1960), p. 508

Lin C.C., Segel L.A. *Mathematics Applied to Deterministic Problems in the Natural Sciences.* Classics in Applied Mathematics (Philadelphia, SIAM, 1988) *[This book which seems to have become a classic 'in its own lifetime' covers a wide range of interesting generic problems.]*

Linderholm C.E. *Mathematics Made Difficult* (London, Wolfe Publishing, 1972)

Luenberger D.G. *Optimization by Vector Space Methods* (New York, John Wiley, 1969)

Lunze J. *Feedback Control of Large Scale Systems.* International Series in Systems and Control Engineering (New York, Prentice Hall, 1991)

Maciejowski J.M. *Predictive Control: With Constraints* (Harlow, Prentice-Hall, 2001)

Maclane S., Birkhoff G. *Algebra* (London, Collier–Macmillan, 1967)

Maddock R.J. *Poles and Zeros in Electrical and Control Engineering* (London, Holt, Rinehart and Winston, 1982) *[Still useful as a source of simply understood examples.]*

Mamdami E.H. 'Applications of fuzzy algorithms for control of simple dynamic plant'. *Proceedings of the IEEE*. 1976;**121**:1585–8

Markus L., Lee E.B. *Foundations of the Theory of Optimal Control* (New York, John Wiley, 1967)

Marlin T.E., Brisk M.L., Barton G.W., Perkins J.D. 'Experiences from an industry–university study on control benefits'. *Preprints of Control Systems Conference Quantifying the Benefits of Process Control*, 1–4 May 2000, pp. 5–9

Mathematical Tables: Laplace Transforms. Mobile Reference (Kindle Edition, 2009)

Matausek M.R., Macic A.D., Dacic D.B. 'Modified internal model control approach to the design and tuning of linear digital controllers'. *International Journal of Systems Science*. 2002;**33**(1):67–79

Matlab Documentation. *Application of H-Infinity and Mu to Active Suspension Control*. R2011a Documentation, Robust Control Toolbox Matlab, 2011

Maxwell J.C. 'On governors'. *Proceedings of the Royal Society of London*. 1868;**16**:270–83

McCollum P.A., Brown B. *Laplace Transform Tables and Theorems* (New York, Holt, Rinehart and Winston, 1965)

McLean D. *Automatic Flight Control Systems* (New York, Prentice Hall, 1990) *[Gives many interesting applications.]*

Mendenhall W., Beaver R.J., Beaver B.M. *Introduction to Probability and Statistics*. 13th edn (Belmont, CA, Duxbury Press, 2008)

Milhoub M. 'Real-time application of discrete second order sliding mode control to a chemical reactor'. *Control Engineering Practice*. 2009; **17**(9):1089–95

Milnor J.W. *Topology from the Differentiable Viewpoint* (Princeton, NJ Princeton University Press, 1997)

Minorsky N. *Introduction to Non-Linear Mechanics* (Ann Arbor, MI, J.W. Edwards, 1947)

Misawa E.A. *Fundamentals of Sliding Mode Control* (New York, John Wiley, 2012)

Mishkin E., Braun L. *Adaptive Control Systems* (New York, McGraw-Hill, 1961)

Mo S.-F., Billingsley J. 'Fast-model predictive control of multivariable systems'. *IEE Proceedings, Part D: Control Theory and Applications*. 1990; **137**(6):364–6

Mohtadi C. 'Generalised predictive control'. *IEE Colloquium (Digest)*. 1987;**33**:1–9 *[IEE Fourth Workshop on Self-Tuning and Adaptive Control, Oxford.]*

Moler C. '19 Dubious ways to calculate the exponential of a matrix'. *SIAM Review*. 1978;**204**:801–37 *[Anyone who sets out to perform the calculation from first principles or who wants to understand what is involved will appreciate this paper.]*

Monshizadeh-Naini N. 'Input−output pairing using effective relative energy array'. *Industrial & Engineering Chemistry Research*. 2009;48(15):7137–44

Moore R.E. *Computational Functional Analysis* (Chichester, Ellis Horwood, 1985)

Morari M., Zafiriou E. *Robust Process Control* (New York, Prentice Hall, 1989)

Mostow G.D., Sampson J.H., Meyer J. *Fundamental Structures of Algebra* (New York, McGraw-Hill, 1963)

Naylor A.W., Sell G.R. *Linear Operator Theory in Engineering and Science* (New York, Holt, Rinehart and Winston, 1971)

Negretti and Zambra 'Automatic temperature recorder controller'. *Journal of Scientific Instruments.* 1934;**11**:259 *[The venerable instrument company Negretti and Zambra was founded in 1850, based initially in London, later in Aylesbury, and manufactured many (now collectable) beautifully made devices for industrial measurement and control.]*

Nemitskii V.V., Stepanov A. *Qualitative Theory of Ordinary Differential Equations* (Princeton, NJ, Princeton University Press, 1960)

Nevalinna R., Paatero V. *Introduction to Complex Analysis* (Providence, RI, AMS Chelsea Publishing, 2007)

Newton G.C., Gould L.A., Kaiser J.F. *Analytical Design of Linear Feedback Controls* (New York, John Wiley, 1957)

Noh J.S., Lee G.H. 'Position control of a mobile inverted pendulum system using radial basis function networks'. *International Journal of Control, Automation and Systems*. 2010;**8**(1):157–62

Nyquist H. 'Regeneration theory'. *Bell System Technical Journal*. 1932; **11**:126–47

Obinata G., Anderson B.D.O. *Model Reduction for Control System Design* (New York, Springer, 2000)

Ogata K. *Discrete Time Control Systems* (Pearson Education, 2006)

Ogata K. *Modern Control Engineering Plus MATLAB & Simulink, Student Version* (Pearson Education, 2010)

Oldenbourg R.C., Sartorius H. *Dynamics of Automatic Control* (New York, ASME, 1948)

Omatu S., Seinfeld J.H. *Distributed Parameter Systems: Theory and Applications* (New York, Oxford University Press, 1989)

Padma Sreea R. Chidambarama M. 'Set point weighted PID controllers for unstable systems'. *Chemical Engineering Communications*. 2005;**192**(1):1–13

Pallu de la Barriere R. *Optimal Control Theory* (New York, Dover Publications, 1967) *[It was an encounter with this book that was the initial motivator for the author to become interested in the operator-theoretic foundations of linear control theory.]*

Papoulis A. *Probability, Random Variables, and Stochastic Processes* (Boston, McGraw-Hill, 2002)

Papoulis A., Angelo E.J. *Pole-Zero Patterns in the Analysis and Design of Low-Order Systems* (New York, McGraw-Hill, 1964)

Park B.J., Hong S.K., Huh C.-D. 'Time efficient input shaping control of container crane systems'. *Proceedings of IEEE Conference on Control Applications*; Anchorage, AK, Sep 2000, vol. 1, pp. 80–5

Park U.-H., Lee J.-W., Lim B.-D., Sung Y.-G. 'Design and sensitivity analysis of an input shaping filter in the z-plane'. *Journal of Sound and Vibration*. 2001;**243**(1):157–71

Parrish J.R., Brosilow C.B. 'Inferential control applied to industrial autoclaves'. *Conference on Real Time Digital Control Applications. Proceedings of the IFAC/IFIP Symposium (code 04517), Guadalajara, Mex (IFAC Proceedings Series 1)*. Oxford, Pergamon Press, 1984

Perruquetti W., Barbot J.-P. (eds.). *Sliding Mode Control in Engineering* (New York, Marcel Dekker, 2002)

Perez T. *Ship Motion Control: Course Keeping and Roll Stabilisation Using Rudder and Fins*. Advances in Industrial Control (London, Springer, 2005)

Pierre C., Perkins N.C. (eds.). Structural dynamics of large scale and complex systems. *American Society of Mechanical Engineers*, 1993 presented at the 1993 ASME Design Technical Conferences, 14th Biennial Conference on Mechanical Vibration and Noise, Albuquerque, New Mexico, September 19–22, 1993

Poincare H. *Les méthodes nouvelles de la méchanique celeste*. Vol. 1 (Paris, Gauthier-Villars, 1892)

Pontryagin L.S., Boltyanskii V.G., Gamkrelidze R.V., Mishchenko E.F. *The Mathematical Theory of Optimal Processes* (Oxford, Pergamon Press, 1964)

Porter W.A. *Modern Foundations of Systems Engineering* (New York, Macmillan, 1966)

Poston T., Stewart I.N. *Taylor Expansions and Catastrophes* (London, Pitman, 1976) *[Contains useful material on the inverse function theorem.]*

Prasad R. 'Padé type model order reduction for multivariable systems using Routh approximation'. *Computers and Electrical Engineering*. 2000; **26**(6):445–59

Prigogine I. *From Being to Becoming* (New York, W.H. Freeman and Company, 1980)

Protter M.H., Morrey C.B. *Analytic Geometry*. 2d edn (Reading, MA, Addison-Wesley Pub. Co., 1975)

Prudnikov A.P., Brychkov Yu.A., Marichev O. *Integrals and Series, Vol. 5: Inverse Laplace Transforms* (New York, Gordon and Breach, 1992)

Purdue University, Rice University, Florida State University, Universit catholique de Louvain. *Model reduction of dynamical systems for real-time control*. Available from http://www.caam.rice.edu/

Qu Z. *Cooperative Control of Dynamical Systems: Applications to Autonomous Vehicles* (London, Springer, 2009)

Ragazzini J.R., Zadeh L.H. 'The analysis of sampled data systems'. *AIEE Transactions*. 1952;71(II):225–34

Rao S.P., Sen I. 'Robust pole placement stabilizer design using linear matrix inequalities'. *IEEE Transactions on Power Systems*. 2000;**I5**:313

Reeves C.R. *Genetic Algorithms: a Guide to GA Theory* (Dordrecht, Kluwer, 2002)

Reynolds T.S. *Stronger than a Hundred Men: a History of the Vertical Water Wheel* (Baltimore, Johns Hopkins University Press, 1983)

Riccati J.F. Animadversiones In Aequationes Differentiates, Acta Eruditorum Lipsiae. Re-printed by Bittanti S. (ed.). *Count Riccati and the Early Days of the Riccati Equation* (Bologna, Milano, Pitagora Editrice, 1724)

[The three papers cited below are directly related to the history of the Riccati equation, which has returned to prominence after two centuries of relative obscurity with the arrival of powerful optimisation synthesis software.]

Bernoulli D. 'SolutioproblematisRiccatianipropositi'. *ASct. Lips. Suppl. Tom. VIII,* p.73. *ActaEruditorum*. 1725:473–75

Bittanti S., Laub A., Willems, J. *The Riccati Equation* (Berlin, Springer-Verlag, 1991)

Bittanti S. 'History and prehistory of the Riccati equation'. *Session Origins of the calculus of variations and optimal control, IEEE Conference on Decision and Control, Kobe, Japan*; December 1996 *[300 years after the publication of Bernoulli's paper on the 'brachystochrone problem' in the ActaEroditorumLipsae (1696).]*

Richalet J., Rault A., Testud J.L., Papon J. *Model Algorithmic Control of Industrial Processes* (Amsterdam, North Holland, 1977) *[Professor Richalet's team based at GERBIOS near Paris, successfully applied the model algorithmic control approach to several high-profile industrial applications.]*

Riesz F., Nagy B.S. *Functional Analysis* (New York, Frederick Ungar, 1971)

Riley K.F., Hobson M.P., Bence S.J. *Mathematical Methods for Physics and Engineering* (Cambridge, Cambridge University Press, 2006) *[Admirably covers most of the mathematics needed by undergraduate physicists and engineers.]*

Roots W.K. *Fundamentals of Temperature Control* (New York, Academic Press, 1969)

Rosen R. *Anticipatory Systems* (Oxford, Pergamon Press, 1985) *[Rosen dares to challenge the universality of the principle of causality.]*

Rosenbrock H.H., Munro N. *Alternatives for Linear Multivariable Control! The Inverse Nyquist Array Method* (Chicago, National Engineering Consortium, 1978), pp. 101–37 *[Describes pioneering work that uses Gershgorin circles to model the effects of loop interactions.]*

Rosenbrock H.H., Storey C. *Computational Techniques for Chemical Engineers* (Oxford, Pergamon Press, 1966)

Rosenbrock H.H., Storey C. *Mathematics of Dynamical Systems* (New York, Wiley Interscience Division, 1970)

Routh E.J. *A Treatise on the Stability of a Given State of Motion* (London, Macmillan, 1877)

Routh E.J. *Advanced Part of the Dynamics of a System of Rigid Bodies* (London, Macmillan, 1930)

Rubio J.E. *The Theory of Linear Systems* (New York, Academic Press, 1971) *[Function-analytic viewpoint.]*

Sabanovic A., Fridman L., Spurgeon S.K. Variable Structure Systems: From Principles to Implementation (London, IET Press, 2004)

Sahinkaya M.N. 'Input shaping for vibration-free positioning of flexible systems'. *Proceedings of the Institution of Mechanical Engineers. Part I: Journal of Systems and Control Engineering*. 2001;**215**(5):467–81

Sage A.P., White C.C. *Optimum Systems Control* (Englewood Cliffs, NJ, Prentice-Hall, 1977)

Saltelli A., Chan K.E., Scott M. (eds.). *Sensitivity Analysis* (Chichester, Wiley, 2000)

Sandefur J. *Elementary Mathematical Modelling: A Dynamic Approach* (Pacific Grove, CA, Brooks Cole, 2000)

Saridis J.S. *Entropy in Control Engineering* (Singapore; River Edge, N.J., World Scientific, 2001)

Seborg D.E., Edgar T.F., Mellichamp D.A. *Process Dynamics and Control* (Hoboken, NJ, Wiley, 2003)

Segel L.A. *Mathematical Models in Molecular and Cellular Biology* (Cambridge, Cambridge University Press, 1980)

Segre E. *From Falling Bodies to Radio Waves* (New York, W. H. Freeman and Company, 1984)

Seron M.M., Braslavsky J.H., Goodwin G.C. *Fundamental Limitations in Filtering and Control.* Communications and Control Engineering Series (London, Springer, 1997)

Shannon C.E., Weaver W. *The Mathematical Theory of Communication* (Urbana, University of Illinois Press, 1972) *[Explains quantification of information in a brilliant short book.]*

Shawe-Taylor J., Cristianini N. *Kernel Methods for Pattern Analysis* (Cambridge, Cambridge University Press, 2004)

Shinskey F.G. *Process Control Systems: Application, Design, and Adjustment*. 3rd edn (New York, McGraw-Hill, 1988) [*Highly practical*]

Simon D. *Optimal State Estimation: Kalman, H [Infinity] and Nonlinear Approaches* (New York, John Wiley, 2006)

Sims R. 'Gaugemeter for strip mills'. Engineering. 1953;**175**:33

Singh T. *Optimal Reference Shaping for Dynamical Systems* (Boca Raton, Taylor & Francis, 2010) *[Explains its subject from first principles.]*

Sivanandam S.N. *Introduction to Neural Networks using MATLAB 6.0* (New Delhi, Tata McGraw-Hill, 2006)

Slone R.D., Lee J.-F., Lee R. 'A comparison of some model order reduction techniques'. *Electromagnetics*. 2002;**22**(4):275–89

Smale S., Williams R.F. 'The qualitative analysis of a difference equation of population growth'. *Journal of Mathematical Biology*.1976;**3**:1–4 *[An early reference describing chaotic phenomena.]*

Smith J.M. The importance of the nervous system in the evolution of animal flight. Evolution. 1952;**6**(1):127–129

Söderstrom T. *Discrete-Time Stochastic Systems: Estimation and Control*. 2nd edn (New York, Springer, 2002)

Söderstrom T., Stoica P. *System Identification* (New York, Prentice-Hall, 1989)

Soeterboek R. *Predictive Control: A Unified Approach* (New York, Prentice-Hall, 1992)

Sommerfeld A. *Partial Differential Equations* (New York, Academic Press, 1949)

Sommerfeld A. *Mechanics of Deformable Bodies*. Lectures on Theoretical Physics. Vol. 2 (New York, Academic Press, 1950)

Sommerfeld A. *Mechanics*. Lectures on Theoretical Physics. Vol. 1 (New York, Academic Press, 1952) *[Three masterly books being examples of the writings of Arnold Sommerfeld; he was active in Munich at about the same time that David Hilbert, based in Göttingen, was altering the face of 20th-century mathematics.]*

Spivak M. *Calculus on Manifolds* (Menlo Park, CA, W.A. Benjamin, 1965)

Stark L. *Neurological Control Systems* (New York, Plenum, 1968)

Steen L.A., Seebach J.A. *Counter-Examples in Topology* (New York, Holt, Rinehart and Winston, 1970)

Sussmann H., Jurdjevic V. 'Controllability of nonlinear systems'. *J. Differential Equations*. 1972;**12**:95–116

Takahashi Y., Rabins M.J., Auslander D.M. *Control and Dynamic Systems* (Reading, MA, Addison-Wesley, 1970)

Tan K.C. (ed.). *Advances in Swarm Intelligence*. Lecture Notes in Computer Science, Vol. 6145 (Heidelberg, Springer, 2010)

Tan N., Atherton D.P. 'Stability and performance analysis in an uncertain world'. *Computing & Control Engineering Journal*. 2000;**11**(2):91–101

Terano T. *Agent-Based Simulation: From Modeling Methodologies to Real-World* (Berlin; Heidelberg, Springer, 2005)

Thaler G.J., Brown R.G. *Servomechanism Analysis* (New York, McGraw-Hill, 1953)

Todhunter I. *Theory of Equations* (London, Macmillan, 1904)

Toraldo Di Francia G. *The Investigation of the Physical World* (Cambridge, Cambridge University Press, 1981) *[Interesting, although only peripherally relevant.]*

Toth R. *Modeling and Identification of Linear Parameter-Varying Systems*. Lecture Notes in Control and Information Sciences, Vol. 403 (Berlin, Springer, 2010)

Tou J.T. *Digital and Sampled-Data Control Systems* (New York, McGraw-Hill, 1959)

Tou J.T. *Modern Control Theory* (New York, McGraw-Hill, 1964)

Truesdell C. *An Idiot's Fugitive Essays on Science* (New York, Springer-Verlag, 1984) *[A provocative book that is relevant in a broad sense: for instance, it compares unfavourably the present-day data-rich, theory-weak approaches to science with the methods of the intellectual giants of the past.]*

Truxal J.G. *Automatic Feedback Control Systems Synthesis* (New York, McGraw-Hill, 1955)

Tsang T.T.C., Clarke D.W. 'Generalised predictive control with input constraints'. *IEE Proceedings, Part D: Control Theory and Applications*. 1988; **135**(6):451-60

Tsien H.S. *Engineering Cybernetics* (New York, McGraw-Hill, 1954) *[Contains otherwise hard-to-find simple but valuable material about relay control systems.]*

Tsypkin Ya. Z. *Adaptation and Learning in Automatic Systems* (New York, Academic Press, 1971)

Tsypkin Ya. Z. *Foundations of the Theory of Learning Systems* (New York, Academic Press, 1973)

Turnbull H.W. *Theory of Equations*. University Mathematical Texts (Edinburgh, Oliver and Boyd, 1963)

Turner M.C., Bates D.G. *Mathematical Methods for Robust and Nonlinear Control*. EPSRC Summer School, Lecture Notes in Control and Information Sciences (Berlin; London, Springer, 2007)

Utkin V.A. 'Variable structure systems with sliding modes'. *IEEE Transactions on Automatic Control.* 1977;**AC-22**(2):212–22

Van der Pol B. 'Forced oscillations in a circuit with nonlinear resistance'. *London, Edinburgh and Dublin Philosophical Magazine and Journal of Science*. 1927;**3**:65–80 *[Pioneering analysis of limit cycle formation in certain types of nonlinear second-order systems.*]

Van der Pol B., Bremmer H. *Operational Calculus Based on the Two-Sided Laplace Integral* (Cambridge, Cambridge University Press, 1955)

Vidyasagar A. *A Theory of Learning and Generalisation: With Applications to Neural Networks and Control Systems*. Communications and Control Engineering Series (London, Springer, 2002)

Vinnicombe G. *Uncertainty and Feedback: H Infinity Loop-Shaping and the V-Gap Metric* (London, Imperial College Press, 2000)

Wang L. *Model Predictive Control System Design and Implementation Using MATLAB.* Advances in Industrial Control (New York; London, Springer, 2009)

Wang L., Cluett W.R. *From Plant Data to Process Control: Ideas for Process Identification and PID Design* (London, Taylor & Francis, 2000)

Wang Q.-G., Lee T.H., Lin C. *Relay Feedback: Analysis, Identification, and Control* (London, Springer, 2003)

Wellstead P.E. *Frequency Response Analysis*. Reports 010/1983 and 2003, Solartron Instruments. Available from http://www.control-systems-principles.co.uk/.../frequency-response-analysis2.pdf *[A very practical paper.]*

Widder D.V. *The Heat Equation* (New York, Academic Press, 1975) *[Included because of its approach to a distributed parameter problem.]*

Wiener N. *The Interpolation Extrapolation and Smoothing of Stationary Time Series* (New York, John Wiley, 1948)

Wiener N. *Cybernetics* (Cambridge, MA, MIT Press, 1961)

Wigner E. *The Unreasonable Effectiveness of Mathematics in the Natural Sciences.* Communications in Pure and Applied Mathematics. Vol. 13, no. 2 (New York, John Wiley, 1960)

Willems J.L. *Stability Theory of Dynamical Systems* (London, Thomas Nelson, 1970)

Winfree A.T. *The Geometry of Biological Time* (New York, Springer-Verlag, 1990) *[What time is it at the North Pole, which belongs to all time zones simultaneously? Are there (amphidromic) points in the oceans where the depth remains constant while the water pivots around like a rotating tilted plane? We live on one*

of several interacting spinning spheres in a world dominated by yearly, monthly and daily cycles. Additionally, all living organisms contain important biochemical clocks. The book considers all these aspects and more; from a deep but understandable geometric viewpoint.]

Winter D.A. Human balance and posture control during standing and walking. Gait & Posture. 1995;**3**(4):193–214

Wonham W.M. 'Geometric state-space theory in linear multivariable control: a status report'. *Automatica*. 1979;**15**:5–13

Wonham W.M. *Linear Multivariable Control: A Geometric Approach.* Applications of Mathematics Series. 3rd edn (New York, Springer, 1985) *['The geometry was first brought in out of revulsion against the orgy of matrix manipulation that linear control theory mainly consisted of not so long ago. But secondly and of greater interest, the geometric setting rather quickly suggested new methods of attacking synthesis that have proved to be intuitive and economical; they are also easily reduced to matrix arithmetic as soon as you want to compute.' (from Wonham's sleeve note)]*

Xu Q. 'Data-driven LQG benchmaking for economic performance assessment of advanced process control systems'. Paper FrC07.6.ACC, San Francisco, 2011

Yaniv O. *Quantitative Feedback Design of Linear and Nonlinear Control Systems* (Dordrecht, Kluwer Academic Publishers, 1999)

Zadeh L.A., Desoer C.A. *Linear System Theory: The State Space Approach* (New York, McGraw-Hill, 1963) *[This is a rigorous, concise, indispensable book.]*

Zadeh L.A. *Toward a Theory of Fuzzy Systems* (Springfield, VA, National Aeronautics and Space Administration, 1969)

Zadeh L.A. 'A rationale for fuzzy control'. *Journal of Dynamic Systems, Measurement and Control*. 1972;**94**(Series G):3–4

Zak S.H. *Systems and Control* (Oxford, Oxford University Press, 2003)

Zakian V. (ed.). *Control Systems Design: A New Framework* (London, Springer, 2005) *[Dr Zakian pioneered his 'method of inequalities' and created a group at UMIST who have continued to develop that methodology, and in this reference, they demonstrate the power of the approach.]*

Zames G. 'On the input-output stability of nonlinear time-varying feedback systems, Pts. I and II'. *IEEE Transactions on Automatic Control*. 1966;**AC-11** (2):228–38, (3):465–77

Zames G. 'Feedback and complexity, special plenary lecture addendum'. *IEEE Conference, Decision and Control*; Clearwater, FL, 1976 *[Zames is usually credited with creating the initial concept from which* H_∞ theory developed.]

Zames G. 'Optimal sensitivity and feedback: weighted semi-norms, approximate inverses, and plant invariant schemes'. *Proceedings of the Allerton Conference*; Urbana, Illinois, 1979

Zames G. 'Feedback and optimal sensitivity: model reference transformations, multiplicative semi-norms, and approximate inverses'. *IEEE Transactions on Automatic Control*. 1981;**AC-26**:301–20

Zhao W.B., Zhenfan L.T. *et al.* 'Neural network based online self-learning adaptive PID control'. *Proceedings of the 3rd World Congress on Intelligent Control and Automation (WCICA)*; Hefei, China, 2000, vol. 2, pp. 908–10

Zhihua Q. *Cooperative Control of Dynamical Systems: Applications to Autonomous Vehicles* (London, Springer, 2009)

Zhou K., Doyle J.C., Glover K. *Robust and Optimal Control* (Upper Saddle River, Prentice Hall, 1996) *[A book by some of the researchers who contributed to the creation and development of the topic.]*

Ziegler J.G., Nichols N.Z. 'Optimum settings for automatic controllers'. *Transactions of ASME*. 1942;**64**(11):69 *[Describes two tuning algorithms that are still the basis for almost all current practical three-term controller tuning.]*

Zinober A.S.I. *Deterministic Control of Uncertain Systems* (London, Peter Peregrinus, 1990)

Zito G. *Digital Control Systems: Design, Identification and Implementation* (London, Springer, 2010)

Appendix A

Case histories

Introduction

Few real-world applications involve only the application of a standard repertoire of control techniques and one must avoid being one of the 'six characters in search of an author' community, walking around armed with a non-adjustable spanner seeking suitable problems. In this appendix, I have described a few typical real case histories of simple yet important problems that needed to be solved by custom-devised approaches as parts of overall control tasks.

The short case histories hopefully have a valuable generic message that real control problems often come embedded in and inseparable from a complex and often challenging environment. The non-standard, case-specific difficulties encountered will frequently overshadow the control aspects and may require considerable effort and some non-trivial original thinking before they are solved. The simple examples below are all typical experiences of an industrial control systems development engineer. However, the same generic message, that application-dependent considerations dominate many real control design projects, still applies to complex research projects that need to work in the real world. That is why so many large control projects involve multidisciplinary research teams to bring in the necessary application expertise in, for example, microbiology or aeronautics.

A1 Control of product thickness in a strip rolling mill: from a control point of view this is predominantly a dead time problem

What follows is a description of an industrial application where one of the dominating characteristics of the process to be controlled is significant measurement dead-time, due to the unavoidable siting of a key sensor well downstream of the actuator responsible for correction. This is a pure dead time problem, with the delay being caused by the time for a product to travel from the process to the sensor, often referred to as a transport lag problem.

Figure A1.1 shows the overall layout of the plant.

In a typical hot strip rolling mill, steel slabs are heated to about 1200°C in a pusher furnace where they progress along supported on water-cooled rails until

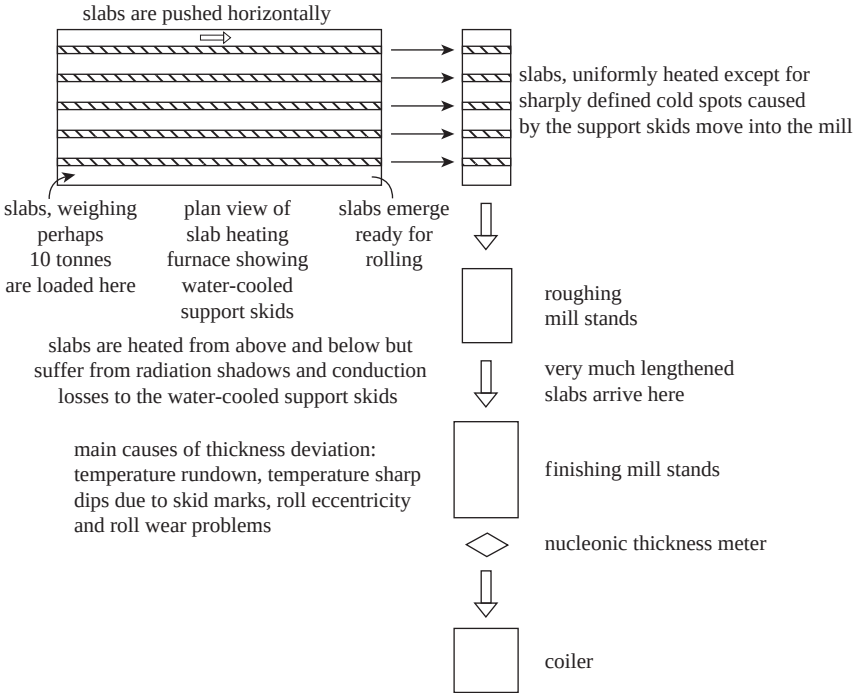


Figure A1.1 The overall layout of a hot strip mill

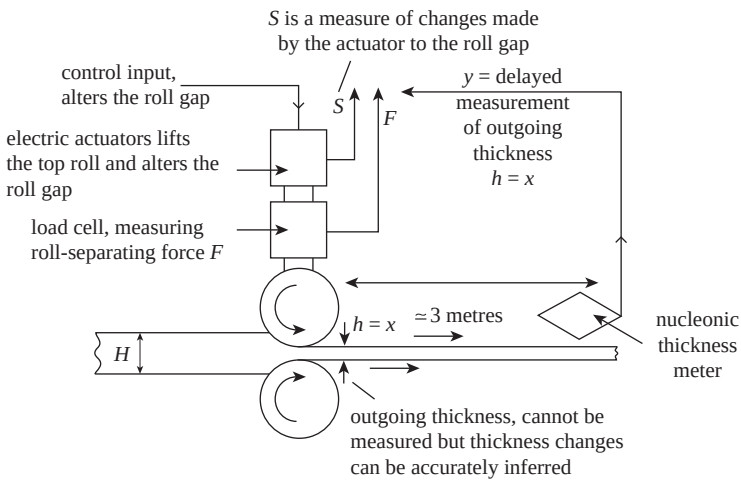


Figure A1.2 A typical control system's view at the end of a strip mill; $h = x$ can be considered to be the unmeasurable state. However, changes in h can be estimated with negligible delay from continuous measurements of separating force F, leaving the measurement y in a calibrating role

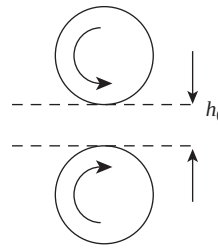


Figure A1.3 The unloaded roll gap h_0. It is the gap between the rolls when no material is being rolled

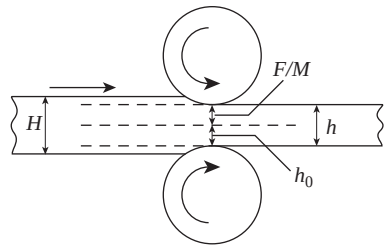


Figure A1.4 The input strip of thickness H is reduced to thickness $h = h_0 + F/M$

being discharged sideways to be rolled lengthways through several sets of rolls designed to achieve the required reduction in thickness. By the time the reduced slab reaches the finishing mill (see Figure A1.2), it is long enough to be in all of the multiple stands of the finishing mill simultaneously, after which it will be turned into a coil by the final machine in the sequence. There are a number of different control problems (temperature, tension, width, speed) to be solved in the configuration. However, here we are considering the achievement of the required consistency and correctness of the final thickness '*h*' of the finished strip. This can be controlled by actuators causing vertical movement of the upper rolls to alter the roll gap (h_0 in Figure A1.3). An accurate and relatively fast acting thickness measurement *y* is available from a nucleonic thickness meter, but the hostile location means that this sophisticated device must be mounted some metres downstream from the rolls that are the thickness control actuators. In other words, the measurement is delayed significantly and this application has to be classified as predominantly a dead time control problem.

The gap between the loaded rolls is now greater (Figure A1.4) than h_0 because the large vertical force generated by the squeezing of the hot metal stretches the mill housing by a significant additional amount F/M according to Hooke's law. The outgoing thickness *h* is the same as the roll gap from which it emerged, since there is no elastic recovery to take into account.

The process engineering–based estimation model that was applied in this case was completely successful and in outline was as follows. Roll-separating force *F* can be measured accurately by load cell and the stiffness of the mill housing *M* can

easily be measured simply by forcing the empty rolls progressively together to produce a graph of generated force F against housing stretch. The rest follows by Hooke's law for an elastic medium, i.e. stretch = force/stiffness = F/M.

The unloaded roll gap h_0 is not easily measured, but it can be inferred by making use of the absolute (but delayed) measurement of h from the downstream nucleonic gauge. This leads to a valuable incremental estimator $dh = dF/M$ that can be used to drive a fast-acting roll gap control loop to keep thickness h constant at some desired value h_d. It is the task of the necessarily slow-acting loop that contains the accurate downstream thickness meter to make sure that h_d is the thickness required by the customer. The principle just described was the intellectual property of British Iron and Steel Research Association (BISRA). It was known as the BISRA gauge meter equation (see R. Sims. 'Gaugemeter for strip mills'. *Engineering*. 1953;**175**:33) and was patented and used worldwide for both hot and cold mills, being a major source of income for the BISRA group of laboratories.

We continue this case study to examine the causes of deviations from desired behaviour. We shall find, as in every other real-life application, that none of those causes fit into the neat 'disturbance' categories that are nearly always assumed in control texts.

Causes of thickness deviations

Cause 1

The slabs that will be rolled into long thin strips are heated in pusher furnaces in which they are supported on longitudinal water-cooled skids. When the slabs emerge, they are uniformly heated except that they bear, typically five, sharply defined cold regions caused by loss of heat to the supporting skids. Colder metal being more resistant to deformation, these cold regions will subsequently cause unwanted localised corresponding strip thickness deviations unless they are corrected by very fast acting control actions.

Cause 2

For logistic reasons, heated slabs emerge from a reheating furnace at 90° to the direction in which they will be rolled. That is, they emerge from the reheat furnace sideways on and are then rolled lengthwise through a sequence of perhaps ten sets of rolls. This means that because of natural cooling the head of each progressing slab is always rolled at a higher temperature than the tail. Cooler metal being more resistant to reduction than hotter metal, the result is that, without some form of corrective control effect, there will be a steady thickness increase along the length of the finished strip.

Cause 3

Probably, the next most important cause of thickness deviation is roll eccentricity, i.e. departure from circularity in any of the several rolls involved in determining the final thickness of the strip. Roll eccentricity requires careful treatment, since the usual estimation equation $dh = df/M$, described beneath Figure A1.4 and used

in control systems, assumes perfectly circular rolls so that all deviations *df* in measured roll force are interpreted as variations in strip thickness.

A number of schemes have been suggested to compensate for roll eccentricity. See Choi *et al*. (1994) and Cao and Du (2006) for examples.

A little reading between the lines in this case history will suggest that the control engineer is largely required because of imperfections in the design, engineering or instrumentation of the process. For instance, considering cause 1, why do the skids that support the slabs being heated have to be parallel or indeed water-cooled? (If, e.g. they could be in a zig-zag configuration, then as the slabs progressed horizontally through the furnace, the cooling shadow effect would be distributed along the slab lengths, rather than in five discrete places).

Taking a view of the instrumentation, why cannot accurate strip thickness measurement sensors be developed that are robust enough to be located much nearer to the roll gap?

Considering cause 2, the thickness rundown due to excessive tail-end cooling could largely be removed if the plant was reconfigured so that it was no longer an inline sequence, but that half way along, slabs were moved sideways and then rolled backwards so that heads became tails in a shorter wider building.

This situation (that most control loops are required because of non-ideal designs, non-ideally consistent raw materials etc.) is entirely typical of industrial processes and leads to a useful view of considering trade-offs between alternative ways of achieving the best overall design of a new plant with control being just one of several different ingredients.

Of course, the process is designed to produce a product and the overall building and operating costs are best minimised by a judicious mix of components. Control expertise needs to be involved from the start in the design process; if not, expensive modifications might need to be made later, for such simple needs as sensor access, that could have been built in, 'free of charge', had it been suggested early enough.

It is hoped that this particular case history has demonstrated a valuable general principal that in-depth process and engineering knowledge as well as knowledge of control-theoretic techniques is absolutely essential to success in real-world applications.

A2 The cut-up problem

This is a control problem squarely within the work of an industrial control engineer. It is mathematically trivial, financially very important and needed the invention of an ingenious original diagram as a framework for its numerical solution.

One of the simplest control tasks in the steel industry is to design a system to cut up, with minimum waste, a long piece of hot metal of length L into shorter lengths as required by the customer, using a so-called flying shear as the cutting agent. The shear is essentially a pair of rotatable rollers that operate in synchronism to cut the fast-moving hot metal into whatever lengths are specified (Figure A2.1).

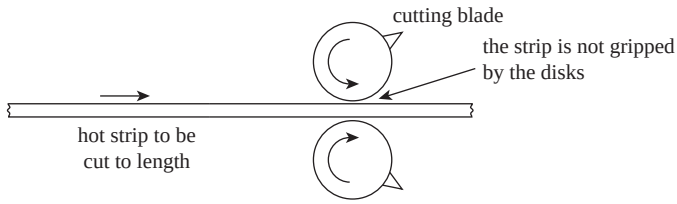


Figure A2.1 A flying shear

This case history relates how the problem to be solved was not only to design the flying shear control loop, which though needing rigorous specification was routine, but also to advise management the best mean length L that would allow optimal cut-up with minimum waste. Imagine for a moment that the customer will accept any length of material in the range [35,36] length units and that the total length L to be cut is in the range $L\in$ [350,360] length units. Then it is obvious that the total length L can be cut into ten equal lengths that meet the customer's requirement leaving zero waste. However, suppose now that L falls into the interval [361,384], then no choice of product lengths for the customer exists that provides, even theoretically, zero waste.

Two things become clear:

(a) As the total length L to be cut up increases, simple arithmetic shows that the width of the solution windows (where ideal cut-up can be achieved) increases until the window $L\in$ [1225,1260] is reached. This solution window is followed and merges into the next solution window $L\in$ [1260,1296], meaning that once L attains or exceeds a length $L = 1225$, there will always be an ideal cut-up solution.
(b) In this application, assume we have constraint $L < 350$ imposed on the total length and observe from Figure A2.2 that, assuming a Gaussian distribution of expected total length L, how viciously asymmetrical the cut-up problem really is. We shall need to advise management to place the target value for L, towards the upper end of the solution interval [315,324], that I have chosen for this illustration.

Any particular total length L with any particular variance can be superimposed and the inevitable losses will be shown graphically and can be calculated numerically.

Two assumed Gaussian distributions for sigma values 3 and 6 (length variation) are drawn on a rebased horizontal axis for convenience. Zero represents a length of 288 units and the solution window [27,36] in the diagram is [315,324] units in reality and the specimen mean length of 32, chosen only for illustration purposes, is in reality 320 units. Superimposed on the diagram are two isosceles right-angled triangles that show the scrappage that will occur for any chosen total length.

To complete the problem we compute the expected loss areas as the Gaussian curve intersect the ends of the solution window.

The Gaussian equation is

$$f(x) = \frac{1}{(\sqrt{2\pi})\delta} e^{-(x-\mu)^2/2\delta^2}$$

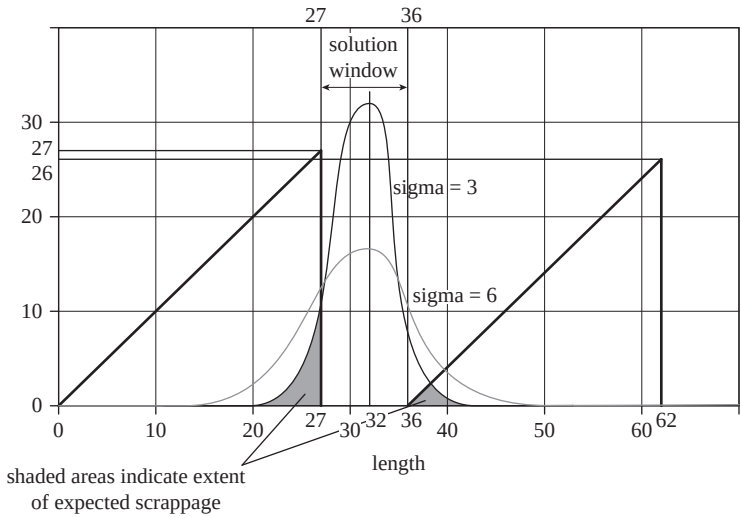


Figure A2.2 *This ingenious visualisation framework was devised by my United Steels Company colleague, Mr Ron Pipe. It shows the solution window where perfect no-wastage cut-up is theoretically possible as the gap between two isosceles triangles that represent regions of inevitable waste*

where μ is the chosen length L, and x is a particular length from the Gaussian population. Rebasing the x-axis so that new $x = 0$ is now at old $x = 288$ as in Figure A2.2, we obtain the expression to be minimised with respect to μ as

$$J = \int_{x=0}^{x=27} xf(d)\,dx + \int_{x=36}^{x=62} (x - 36)f(x)\,dx$$

The results are shown in the following table in terms of expected length of wastage per cut-up operation:

L	$\partial = 3$	$\partial = 6$
27	**12.30432**	**39.30432**
28	**9.215753**	**37.21575**
29	**6.373898**	**35.3739**
30	**4.059218**	**34.05922**
31	**2.394997**	**33.395**
32	**1.358045**	**33.35805**
33	**0.838728**	**33.83873**
34	**0.708414**	**34.70841**
35	**0.862584**	**35.86258**
36	**1.232128**	**37.23213**

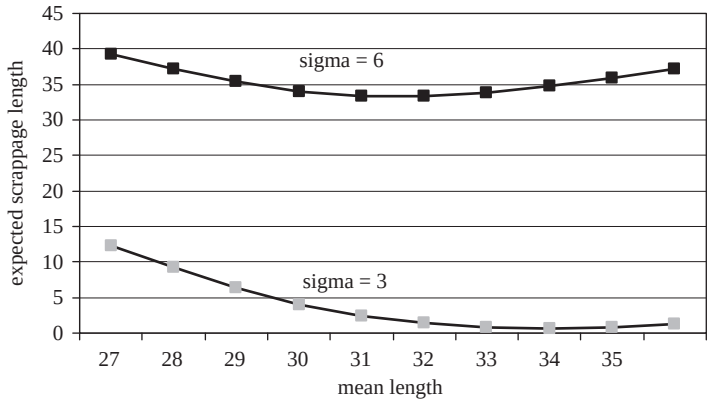


Figure A2.3 Expected scrappage as a function of mean length to be cut up

Figure A2.3 plots the same results and it is clear that the choice of optimum length L is ∂ dependent, with recommended choices to management for L being $L = 34$ for $\partial = 3$ and $L = 32$ for $\partial = 6$.

A3 Control of the pressure inside small fuel-fired furnaces

The problem to be described here is typical of that occurring in small (a few hundred employees) companies.

It is important to remember that these are small unsophisticated pieces of equipment usually operated by companies unable to commit to some of the available advanced furnace design features that could be used to 'design out' most of the problems that are about to be described.

Cold billets are presented to a pusher device at the right door of the furnace and the material progresses right to left, being taken from the left door as called for by a small rolling mill at temperatures of, typically, 1100°C. The doors are frequently opened for operational reasons and that causes a significant control problem.

There is a very strong buoyancy effect because of the elevated temperatures inside the furnace, and without some form of pressure control, unwanted air will be sucked through the doors every time they are opened and will cool the slabs inside.

In a typical installation, as shown in Figure A3.1, a combined pneumatic/hydraulic device measures the furnace pressure through a pipe in the furnace roof, compares it with a desired value and then moves a hydraulic piston to move a heavy damper that sets the size of the opening for exit of the hot gases to a chimney.

The desired pressure for the furnace must always be positive to minimise ingress of cold air through the doors and other orifices. However, choice of too high a desired value causes its own problems, just as does the choice of too low a value. In fact, as will be explained, there is a delicate balancing act to be performed.

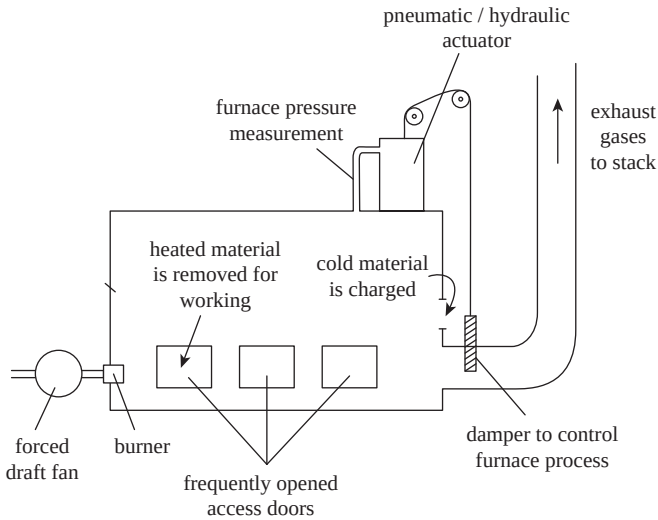


Figure A3.1 A small fuel-fired furnace such as that used to reheat steel slabs for specialist rolling for eventual products such as twist drills

The balancing act occurs because of the peculiar nature of the relation between damper position and resulting furnace pressure (Figure A3.2). There it can be seen that there is a more or less linear operating range where a particular chosen small positive pressure should be able to be achieved and held. What the figure also shows is that, outside that limited 'linear range', there are two 'points of no return' beyond which runaway behaviour exists in both directions as indicated by the sequences of one-way arrows. Perhaps surprising but true, the behaviour at the left of the diagram indicates that opening the dampers beyond a certain point causes a large influx of cold air that, once heated and expanded, increases; the furnace pressure and the dampers move to fully open and stay there.

At the opposite end of the range, beyond the right hand 'point of no return', the situation is analogous. Gases begin to escape through furnace orifices with loss of furnace pressure, rapidly resulting in dampers closing completely and all the products of combustion disastrously entering the workplace.

Additionally, this application suffers from very rapid disturbances (causing large pressure changes over times of a few seconds) that have to be counteracted by slow-moving actuators that may take about 5 s before they start to move. When a typical graph (Figure A3.3) of furnace pressure is examined and the rapidity of wind gusts at the top of the chimney is appreciated and compared with the lumbering response of heavy dampers typically driven by hydraulic cylinders, the overall difficulty of this control problem can be understood.

Furnace pressure control is well known to be difficult, particularly where rapid wind gusts from a chimney can change the internal pressure in a matter of milliseconds. The correcting actuator will often be a heavy cumbersome damper driven

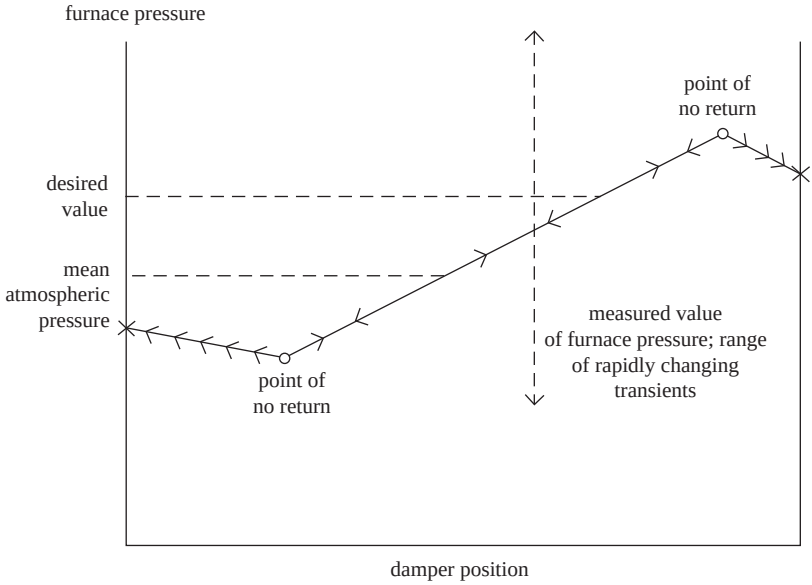


Figure A3.2 Diagrammatic relation between damper position and resulting furnace pressure (note the two 'points of no return')

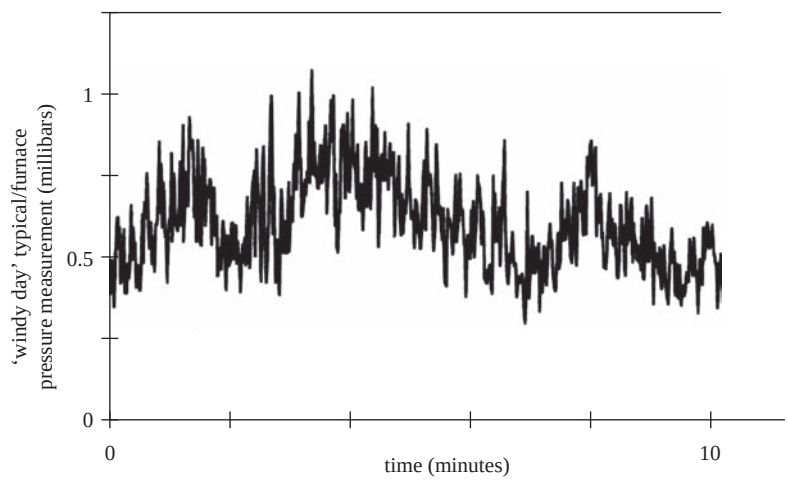


Figure A3.3 A typical furnace pressure graph on a windy day

by a hydraulic cylinder – such a configuration may well take several seconds before even starting to move.

To make life even more difficult, some older furnaces were built with the hearth sloping, so as to assist by gravity, the progress of work pieces along the

furnace. This means that the often poorly fitting access doors are at different heights with the result that cold air may be drawn in at lower levels and simultaneously expelled from doors at higher levels, with the whole unwanted airflow being driven by buoyancy, which at 1000°C is a very significant force.

Suppliers of modern reheating furnaces have 'designed out' most if not all the problems described above, but inherently difficult control problems similar to the one just described will still be encountered in industry by a consultant control engineer.

A4 Batch control: a brief case history of one process: Oxygen Steelmaking

Most of the world's steel is made by the Basic Oxygen Steelmaking (BOS) process. Figure A4.1 shows an outline of the process.

In essence, 300 tonnes of molten iron at about 1200°C and containing about 4% of carbon and many other mostly unwanted elements is poured into a tiltable vessel/reactor. Pure high-pressure oxygen from a water-cooled lance is injected, and after about 20 minutes, steel at up to about 1700°C and containing a desired percentage of carbon (typically less than 0.5%) is produced. The vessel is tilted to allow the hot steel to be poured out and the process repeats. The temperature-gain comes from exothermic oxidation of carbon and other elements, including iron (Figure A4.2). Carbon is reduced to levels required in steelmaking, and undesirable impurities, particularly sulphur and phosphorus, are reduced to acceptable steel-making levels through interactions with the basic (i.e. alkaline) brick lining of the reactor. Figure A4.3 illustrates the asymmetry of the de facto cost function for the Basic Oxygen process in the carbon temperature plane.

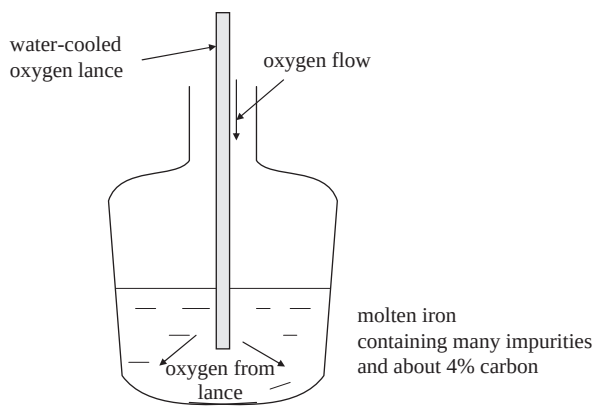


Figure A4.1 The Basic Oxygen Process in outline

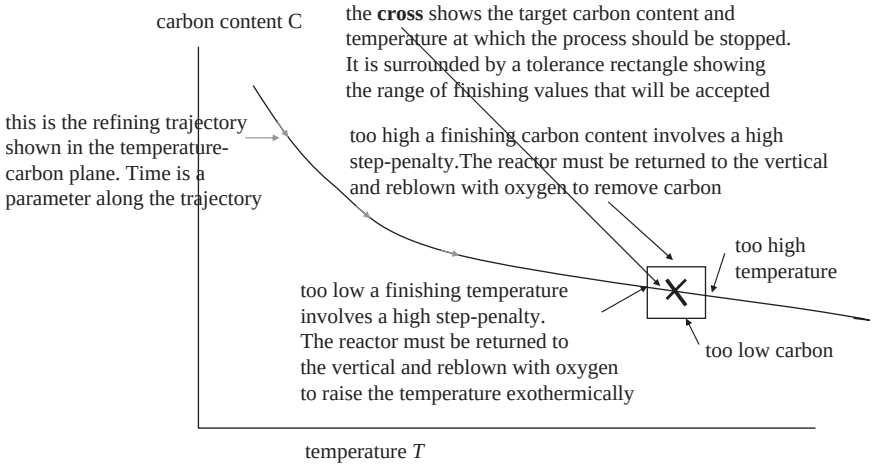


Figure A4.2 The behaviour of the Basic Oxygen Process in the carbon vs. temperature plane. The batch starts with a high carbon content and a low temperature and moves along the trajectory shown as a line in the figure with carbon content reducing and temperature increasing. The aim is to stop the process when carbon and temperature are within the tolerance rectangle shown in the figure. However, the penalties for being outside the rectangle are asymmetrical as is indicated in Figure A4.3

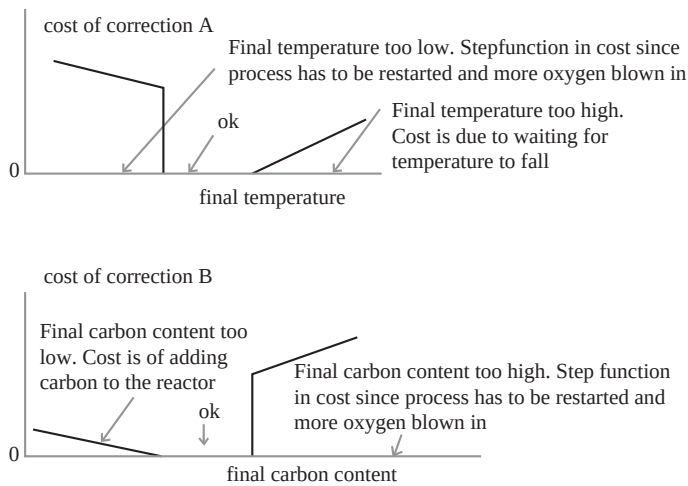


Figure A4.3 An approximation to the de facto cost function at the end of the batch steelmaking process (The total cost of correction if the target rectangle is missed is a combination of the two costs of correction.)

A5 A note on the introduction of novel measurement sensors into control systems

It has been pointed out earlier in this book that the process/system variables that matter most can rarely be measured, online, undelayed, robustly and reliably. However, from time to time, physicists and others come forward to offer a new sensor system that promises to measure an important but previously online-unmeasurable quantity (say α) whose excessive variability is causing significant economic losses. Almost by definition, such sensor systems cannot have been fully tested under closed loop conditions; therefore, should they be put to use as elements of a new closed loop system, designed to reduce the variance of α, then under a law that others must have noticed but this author is calling 'the conservation of variability inside a closed loop', inevitably, some other process variable (say β), inside the control loop will begin to vary significantly for the first time. Such variation, transferred around the control loop, has in the past, significantly degraded the empirical correlations on which proposed new inferential sensors often relied for cases (a) where β was a product property and cases (b) where β was a process variable.

Notation

The notation conforms to 'standard usage' – there are no novel notations. However, the following list, in which symbols are defined with respect to the first chapter in which they appear, may be found useful.

Chapter	Symbol	Meaning
3	G	Operator representing a system to be controlled
	D	Operator representing a controller
	H	Operator representing the behaviour of a composite system
	v	The desired value for y
	y	The measured value of system output
	e	The error between v and y
4	$\mathcal{L}\{\ \}$	The operation of Laplace transforming
	$\mathcal{L}^{-1}\{\ \}$	The operation of inverse Laplace transforming
	s	The complex variable associated with the Laplace transform
	$\mathfrak{F}\{\ \}$	Fourier transformation
	$*$	Convolution
	$R_{u,y}$	The correlation function between u and y
	ζ (zeta)	Damping factor
	ω	Frequency
	ω_n	Undamped natural frequency
	ω_d	Damped frequency
	ω_r	Resonant frequency
	R(P)	Real part of P
	I(P)	Imaginary part of P
	σ (sigma)	The real part of a complex number – often used to label the real axis while $j\omega$ is used to label the imaginary axis
5	$\dot{y}$	dy/dt
	$\ddot{y}$	d^2y/dt^2
6	$\mathbb{R}^n$	n-dimensional real space
	$x(k)$	The value of x after k sampling intervals has elapsed
7	δx	A small perturbation in x
	$\partial u/\partial v$	The partial derivative of u with respect to v
	$x_N(t)$	A nominal trajectory that x is, a priori, expected to follow
8	$\langle , \rangle$	Inner product
	$\|\|x\|\|$	The norm of the vector x
	∇v	The gradient of the scalar v

(*Continues*)

(*Continued*)

Chapter	Symbol	Meaning
10	y^*	The signal y after being sampled
	$z\{\ \}$	The operation of Z transformation
	z	The complex variable associated with the Z transform
	ω_s	Sampling frequency
	G_0	The transfer function of a zero-order hold device
	$G'(s)$	$G_0(s)G(s)$
11	(g_{ij})	The matrix whose typical element is g_{ij}
	Φ	The transition matrix (defined by (11.18))
	Ψ	The matrix defined by (11.18)
	dom L	The domain of L
	ker L	The kernel of L
	dim X	The dimension of the space X
13	[,]	A closed interval
	λ	Lagrange multiplier (do not confuse with usual usage as eigenvalue)
	$\mathcal{R}$	The reachable set
	$\delta\mathcal{R}$	The boundary of the reachable set
	Ω	The admissible set of controls
	$\mathcal{A}$	The attainable set
	sign (x)	$= -1$ if $x < 0$, $= 0$ if $x = 0$, $= 1$ if $x > 0$
	sup	supremum
14	$\hat{x}$	An estimate of x
	$\tilde{x}$	The prediction error x–$\tilde{x}$
	$x(j\|j-1)$	A prediction of the variable $x(j)$ made at time $(j-1)$
	$K(j)$	The Kalman gain at time j
	$\mathcal{E}$	Expected value
15	$H_p, p > 0$	The family of Hardy spaces
	H_∞	The Hardy space of all stable linear time-invariant continuous time system models
	ΔG	A perturbation to a plant transfer function G
	S	The system sensitivity coefficient
	T	The system complementary sensitivity coefficient
	$\sigma_i(A)$	The ith singular value of some matrix A
	$\overline{\sigma}(A)$ $\underline{\sigma}(A)$	The largest and smallest singular value of A, respectively
	$R(A)$	The range space of A
	$N(A)$	The null space of A
	A^*	The adjoint of A
	$\delta_\nu(G_1, G_2)$	The distance between two transfer functions as measured by the ν gap metric
	$b_{G,\ D}$	The distance between a transfer function G and a controller D as measured by the b metric
	$L^p\ [a, b]$	Lebesgue spaces defined on the interval $[a, b]$
	P	The space of all polynomials
	C_n	The space of all n times differentiable functions
	C	The space of all continuous functions
	$\mathcal{C}$	The set of all convergent sequences
	$\mathcal{C}_0$	The set of all sequences convergent to zero
	$l^p, p > 0$	A sequence space

Afterword

Visualisation of the evolution of control design approaches: an overview in a single diagram

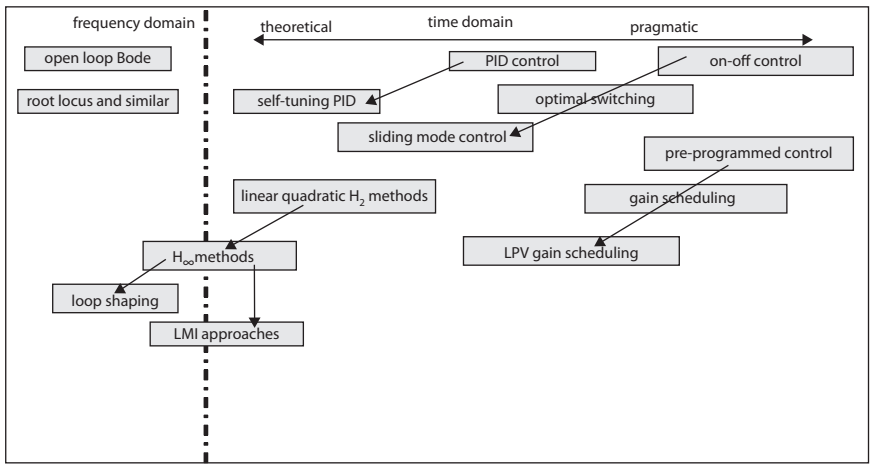


Figure 1 Visualisation of the evolution of control design approaches

Despite its superficial complexity, the overall aim of control design is quite simple: to produce a reliable universal enforcer of behaviour that can impose any given desired behaviour on any process or system whatsoever.

It seems appropriate, in an Afterword, to show how control design methods fit together and have evolved over the years: Figure 1 is the author's attempt. The figure shows how, in a broad sense, design methods have moved from pragmatically-driven towards research-driven while tending to move towards the frequency-domain.

In this diagram, the earliest approaches are at the top; evolution follows the arrows and, in the time domain, has, over time, moved from heavily empirical to more theoretical.

Interestingly, the first broad-based scientific understanding of feedback loops was in the frequency domain (Nyquist, 1932). Now many of the robust control design approaches have returned to the frequency domain, where the constraints on what can be achieved are still determined by exactly the same invariant laws that Bode stated in 1945! See Bode (1945).

Index

Page numbers followed by '*f*' and '*t*' indicate figures and tables respectively.

Adams Prize competition, 370
adaptive and model-based control
  literature resources, 378
adaptive logic networks (ALN), 352
  literature resources, 377
adaptivity
  learning *vs.*, 362
  self-tuning controllers and, 361
additive uncertainty model, 322*f*
AI: *see* artificial intelligence (AI) techniques
aircraft design, stability and, 133–135, 134*f*
Aizerman's conjecture, in envelope methods, 96, 96*f*
algorithm design
  approaches to, 188–192
    direct controller synthesis, 188–189, 189*f*
    gain plus compensation approach, 190–191, 190*f*, 191*f*
  choice, guidelines for, 193
ALN: *see* adaptive logic networks (ALN)
American Control Conferences, 374
ANN: *see* artificial neural networks (ANN)
anticipatory systems, 72–74
artificial GA, 356–358, 357*f*
artificial intelligence (AI) techniques, 343, 365–366
artificial neural networks (ANN), 344–352
  architectures, for dynamic processes, 349–352, 349*f*, 350*f*, 351*f*
  basic building block, 345, 345*f*
  motivation, 344, 344*f*
  multilayer networks, 347, 348*f*
  neural network training, 347–349
  neurons
    architecture of, 345, 345*f*
    properties, demonstration of, 345–347, 346*f*
    properties with *n* inputs, 347
  SOMs for data reduction and clustering, 352
  SVM and ALN, 352
autocorrelation function, 42
automatic control
  defined, 6–7, 7*f*
  schemes for industrial application, methodologies for, 177–182, 178*f*–182*f*
  system
    realisation of, 13*f*
    requirements for, 12–13, 13*f*
automatic feedback control, 14, 14*f*
  loop, synthesis of, 19–28
automatic feedback controllers, 22
autonomous agents and swarms, GA and, 360
axiom 3/axiom 5, visualisation, 204*f*

backlash, in non-linear systems, 95
back-propagation, 347–348
Banach spaces, 339, 372

bang-bang nature of a unique time-optimal control, 283
bang-bang solution, 279
Basic Oxygen Steelmaking (BOS) process
  batch control (case study), 417, 417*f*, 418*f*
basis functions, 89
batch process, control systems, 166–169, 167*f*
  BOS process, 417, 417*f*, 418*f*
  stages, 167
Beale, Guy, 292
Becerra, Victor, Dr., 318
Bell Telephone Laboratories, 165
Bernoulli, Daniel, 285
$b_{G,D}$ metric, 327–328, 327*f*, 329–330
  properties, 328
BISRA: *see* British Iron and Steel Research Association (BISRA)
block manipulation, Laplace transform and, 32–33, 33*f*
Bode, Hendrik, 47, 59–60, 60*f*, 165, 166*f*
Bode criteria, 113
Bode diagram, 48, 49, 49*f*
Bode magnitude plot, 313, 313*f*
Bode plot check on stability, 187
Bode's feedback amplifier, 165–166, 166*f*
Bode's laws, 318–319, 370
Bode's theorem, design constraints by, 144–145
Bohr model, 64
BOS process: *see* Basic Oxygen Steelmaking (BOS) process
boulder, moving
  problem of, 5–6, 5*f*
bounded input/output, 198
British Iron and Steel Research Association (BISRA), 410
calculus of variations, optimal control problem and, 275–276
cancellation, pole, 37
canonical state space representations, 224–236, 224*f*, 225*f*
  advantages, 224–226
  controllability canonical form (phase-variable form), 228–231, 228*f*
  Jordan canonical form, 234–236
  observability canonical form, 231–232, 232*f*
  reachability canonical form, 226–228, 226*f*
  reconstructibility canonical form, 232, 233*f*
  state equations for multi-input, multi-output processes, 233–234
carbon feed rate constraints
  in *Escherichia coli* fermentations, 169, 169*f*
cascade control
  state space view of, 249, 250–254, 250*f*
    equations by inspection from block diagram, 251–253
    jacketed reactor: *see* jacketed reactor
CASE: *see* Computer-aided software engineering (CASE)
catastrophe theory, 108
Cauchy–Riemann conditions, 147
Cauchy's integral theorem, 34
causality, in mathematical modelling, 76–77
centre of gravity (CG), of aircraft, 133, 134*f*
certainty, 71–72
CG: *see* centre of gravity (CG)
chaos, 74–76
chaotic behaviour, 74–76, 75*f*
  non-linear systems and, example, 109–110, 110*f*

Chetaev's theorem, 333
classical control theory
  complex functions, mapping of, 145–147, 145*f*–146*f*
classical techniques, 369
closed loop system
  behaviour, 131, 234, 326, 327
  control, 25*f*, 65, 125*f*, 156*f*, 315*f*, 361
  to desirable locations, poles of, 38–39, 38*f*, 39*f*
  response, 144, 163
  robust stability, 313–314, 314*f*
  self-tuning controller into, three coefficients and, 165
  stability of, 49, 49*f*
  step response of open loop system, 143, 144*f*
  tuning procedure into, 164–165
coefficients
  numerical
    classification, 66
    experimental determination of, 68
    by parameter estimation algorithms, 67
  of three-term controller, 158–165, 158*f*, 161*f*–162*f*, 163*f*
    Ziegler–Nichols equations and, 158–159, 164
commissioning and maintenance, 16, 17*f*
complex eigenvalues
  third order system, modal analysis for, 261–264
  see also eigenvalues
complex function
  derivatives of, 147
  mapping of, 145–147, 145*f*–146*f*
  singularities of, 147–148
complex plane
  pole and zero locations in, 34–35, 34*f*, 35*f*, 36*f*
complex processes, Kalman filter and, 305
complex variable theory, 34
computer-aided software engineering (CASE), 16, 17*f*
computers as system components, 185–187, 186*f*–187*f*
connectedness, 1–2
conservation of dimension, under linear transformations, 245–247
continuous time processes
  under digital control, 187–188, 187*f*
  Kalman filter and, 305
continuous time systems
  controllability, 218
  controllability and observability tests for time-invariant
    by diagonalisation method, 219
    by matrices *Qc*, *Qo*, 218
  modelling through state space equations
    general non-linear model, 202*f*, 205
    input vector, 205
    model linearisation, 205–206
    output vector, 205
    state vector, 205
  observability, 218
  reachability, 218
  reconstructibility, 218
  stability tests for, 217–218
controllability, 247–248
  continuous time systems, 218
  discrete time system, 222–224
  tests
    by diagonalisation method, 219
    by matrices *Qc*, *Qo*, 218
controllability canonical form (phase-variable form), 228–231, 228*f*
controllability Gramian, 248
control laws, 21–22
controller gain, three-term controller coefficients and, 164–165

controllers
control law and, 21–22
in feedback control, 19, 19*f*
non-linear, 22
three-term, 54
controller switching, 360–361
control loop(s)
changes, robust, 124–125, 125*f*
computer, 185–187, 186*f*–187*f*
dead time, 139–141, 140*f*, 141*f*
describing function method for, non-linearities and, 97–98, 97*f*
design, H_∞, 312–314
robust stability of closed loop, 313–314, 314*f*
stability of feedback loop, 312–314, 313*f*
see also H_∞
control problem, defined, 236
control specification, user's requirements and, 175–177, 176*f*
control system design
automatic control in, 6–7, 7*f*
batch process, 166–169, 167*f*
elements for, 4, 5*f*
examples of, 7–9, 8*f*
in frequency response methods, 51, 51*f*
gain scheduling: *see* gain scheduling
implementation and justification, 149–182
industrial application: *see* industrial application, automatic control schemes for
input shaping, 170–172, 171*f*
non-mathematical treatment, 11–17
application of, 14
automatic control system: *see* automatic control, system
automatic feedback control, 14, 14*f*
basic control ideas, 14, 15*f*
broad task, 14, 15*f*
methodology for, 11*f*
overview, 11–12
sequence of objectives, in typical control project, 14, 15*f*
software engineering context, 14, 16–17, 16*f*, 17*f*
system model inversely to synthesise actions, using, 12, 12*f*
rudimentary on-off control, 149–154
sliding mode control, 151–155, 152*f*, 153*f*, 154*f*
three-term controller, 155–166
see also three-term controller
user's requirements into control specification, converting, 175–177, 176*f*
variables, 149
variable structure systems, 151–155
control system(s)
batch process, 166–169, 167*f*, 417, 417*f*, 418*f*
changes, robustness, 124–125, 125*f*
digital, discrete time and: *see* digital control system, discrete time and
discipline, structure for, 370–371, 371*f*
literature resources, 373–383
historical interest, 382
library books, 374
mainstream books, 375
mathematical, 379
older mainstream books, 375
online resources, 374
overview, 373
LMI method: *see* Linear Matrix Inequalities (LMIs) method
non-minimum phase systems, 135–139, 136*f*
see also non-minimum phase systems
performance, limits, 111–148, 121–124, 121*f*–123*f*
Bode's theorem and, 144–145
sensitivity functions and, 141, 143

on synthesis, 126
theoretical, 141–148
time domain, integral constraints in, 143–144, 144*f*
software, 374
stability: *see* stability
stochastic aspects, 378
techniques: *see* control techniques
technology development, review of, 367–370, 367*f*, 368*f*
unstable system, 126–135
see also unstable system
control techniques, 343–366
AI techniques: *see* artificial intelligence (AI) techniques
ANN: *see* artificial neural networks (ANN)
controller switching, gain scheduling and adaptivity, 360–361
fuzzy logic: *see* fuzzy logic
fuzzy set theory: *see* fuzzy set theory
GAs: *see* genetic algorithms (GA)
intelligent systems, 364–365, 364*f*
learning systems with/without supervision, 361–364
overview, 343–344
probing controller, 365–366
control theory
definition, 2, 3–6
Laplace transform in, 31–32, 31*f*
mathematical roots of, 371–372, 372*t*
systems in: *see* systems
convolution integral, 40–41, 40*f*
coprime factorisation uncertainty model, 322, 323*f*
cost calculation, in automation: *see* industrial application, automatic control schemes for
crisp and fuzzy sets, 353, 353*f*
cross-correlation, transfer function by, 41–42
crossover mechanism, 356, 357*f*
cruise (automatic highway speed) control, 329
cut-up problem, 411–414, 412*f*, 413*f*, 414*f*
flying shear, as cutting agent, 411–412, 412*f*

damping
degree of, value of derivative term to control, 156–157, 156*f*, 157*f*
natural frequency, formula for, 43–44, 44*f*
dead time, 139–141, 140*f*, 141*f*
significant, 140–141, 141*f*
transfer function of, 140
ΔG envelope
H_∞ and, 322, 322*f*–323*f*
$\delta\nu$ metric, 327–328, 329–330
derivative of function, as best linear approximation, 107–108, 107*f*, 108*f*
derivative term
to control degree of damping, value of, 156–157, 156*f*, 157*f*
three term controller, 157, 157*f*
describing function method, 95–96
for control loops, non-linearities and, 97–98, 97*f*
determinism, in mathematical modelling, 78
development
of control systems discipline, 370–371, 371*f*
of control technology
drivers for, 369
mathematical optimisation techniques and, 370
phases of, 368–369, 369*f*
problems in, 369
see also control, technology development
of control theory, 371–372, 372*t*

Dewey system, 374
diagonal dominance, concept of, 258
diagonalisation method
  controllability and observability tests for continuous time, time-invariant systems by, 219
  time solutions calculation using transition matrix, 208–211
difference equations, 380
  differential equations to approximating, 193–194
differentiable functions, 100
differential equations
  to approximating difference equations, 193–194
  frequency response from, 54–56
  Laplace transforms for, 31
  transfer function from, 33
differential topology, 380
digital control system, discrete time and, 185–198
  algorithm design: *see* algorithm design, approaches to
  computers as system components, 185–187, 186*f*–187*f*
  discrete time algorithms, 187–188, 187*f*
digital to analogue convertor, 190
dimensional analysis, in mathematical modelling, 85–86
dimensionless frequency, 58
direct controller synthesis, algorithm design, 188–189, 189*f*
discrete time algorithms, 187–188, 187*f*
  defined, 187
discrete time linear system, LMIs method and, 333
discrete time state equation
  existence and uniqueness of solution, 219–220
  with forcing, 221
  transition matrix, 220
  unforced, 219–220
discrete time system
  computers, 185–187, 186*f*–187*f*
  controllability, 222–224
  defined, 185
  digital control system and: *see* digital control system, discrete time and
  observability, 222–224
  reachability, 222–224
  reconstructibility, 222–224
  stability tests, 222
discretisation, 194
distributed systems, modelling of, 87–89
  alternative approaches, 88–89
disturbance rejection, H_∞ design procedure and, 317–318, 318*f*
disturbing signals, Kalman filter and, 305
Dodgson, Charles L., 82
doubt and certainty, mathematical modelling, 71–72
dynamic processes, neural networks architectures for, 349–352, 349*f*, 350*f*, 351*f*
dynamic systems, 2, 6
  transfer function of, 32

economic dynamic programming, 180
economic element, 24, 25*f*
economic justification, of investment in automation, 177–182
  methodologies and illustrations, 177–182, 178*f*–182*f*
eigenvalues
  complex
    state plane diagram for second-order system with, 100, 100*f*
    third order system, modal analysis for, 261–264
  H_∞ norms from dynamic process models, 322–324
    rectangular matrix A, 324
  state plane diagram for second-order linear system with, 99, 99*f*

eigenvectors, trajectories, 99
Einstein, A., 72
Electronic Thesis Online Service (EThOS), 374
envelope methods, 96, 96*f*
equality constants, 277
equations, theory of, 381
equivalent systems, 216
Escherichia coli, 169
  fermentations, carbon feed rate constraints in, 169, 169*f*
EThOS: *see* Electronic Thesis Online Service (EThOS)
Euclidian norm, 282, 282*f*
Euler, Leonard, 285
Euler–Lagrange conditions, 275
European Control Conferences, 374
experimentation
  mathematical modelling and, 64–66
  on plants, 83–85, 83*f*, 84*f*, 85*f*

Fatou's theorem, 339
feasibility stage, LMIs method, 332–333
feedback amplifiers, telephony and, 165–166, 166*f*
feedback control, 5, 236
  see also H_∞
feedback control loops
  synthesis of automatic, 19–28
    alternatives to, 28
    control laws, 21–22
    example, 24–26, 25*f*, 26*f*
    feed-forward control, 28
    heating process, 22–23, 23*f*, 24*f*
    overview, 19–21
    prediction followed by control, 28
    pre-programmed control, 28
    strategies, general conditions for, 27–28, 27*f*
    transient responses ranging from highly oscillatory to sluggish, 26, 27*f*
    working, 22–27
feedback loop design, frequency response in, 50, 50*f*
feedback systems, 2, 3*f*
feed-forward control, feedback control and, 28
Ferrari's method, 115
finite settling time (dead-beat) control, 239
Floquet–Lyapunov transformation, 214
flying shear, as cutting agent, 411–412, 412*f*
forced discrete time state equation, 221
forced pendulum
  non-linear systems and, example, 109–110, 110*f*
Fourier expansion, 98
Fourier transformation, 40
'Fourier type' modelling, 87
frequency, time domains and, 325
frequency response
  Bode diagram, 47, 48, 49, 49*f*, 59–60, 60*f*
  curve, transfer function from, 39
  from differential equation, 54–56
  in feedback loop design, 50, 50*f*
  of linear system, 47–48, 48*f*
  methods, 47–61
    control design in, 51, 51*f*
    design using, 47
    overview, 47
    state space representations and: *see* state space
  Nyquist diagram and Nichols chart, 60–61
  poles and/or zeros, imaginary axis and, 58–59, 59*f*
  practical point, 50
  of second-order system, 57–58, 58*f*
  stability and, 48–49, 49*f*
  for stability margins, 50
  of system, obtaining, 51–53
    difficulties, in practice, 52–53
  testing, system's transfer function and, 56–57, 57*f*
  unit step response of system, 53–54, 53*f*

Froude number, 86
Fuller, Tom, Dr., 382
functional analysis, literature resources, 381
furnace pressure control, 414–417, 415*f*, 416*f*
fuzzy control, literature resources, 377
fuzzy logic, 353–356
  algorithms, 343, 354–355
  characteristics of, 355–356
  literature resources, 377
  references, 356
fuzzy set theory, 353–356
  complex and ill-defined problems, 355
  crude rule of thumb and, 354–355, 355*f*
  introduction and motivation, 353–355, 353*f*, 354*f*
  references, 356

GA: *see* genetic algorithms (GA)
gain, 48, 48*f*
  plus compensation approach, algorithm design, 190–191, 190*f*, 191*f*
gain matrix, optimal, 303–304, 304*f*
gain scheduling, 172–175, 361
  LPV
    approach, outline of, 175
    traditional ad hoc and, 173–175, 174*f*
Galileo, 85
Gaussian equation, 412–413
genetic algorithms (GA), 343, 356–360
  advantages, 359
  artificial, 356–358, 357*f*
    crossover mechanism, 356, 357*f*
    mutation mechanism, 356, 357*f*
  basic ideas, 356
  classical, main steps in, 358–359
  as design tools, 358, 358*f*
  disadvantages, 359
  as general purpose optimisation algorithms, 356, 357*f*
  literature resources, 377–378
  overview, 358–359, 359*f*
  references, 359–360
  rivals to, 360
genetic programming, 343
  literature resources, 377–378
Gifford, John D., 168
global stability, 118, 118*f*
  local and, 118, 118*f*
Google Books, 374
Google Scholar, 374
Guillemin's technique, 40

H_∞, 339
  control loop design, 312–314
    robust stability of closed loop, 313–314, 314*f*
    stability of feedback loop, 312–314, 313*f*
  ΔG envelope, 322, 322*f*–323*f*
  dynamic process models
    eigenvalues of rectangular matrix A, 324
    frequency and time domains, 325
    singular values and eigenvalues, 322–324
    singular values and their relation to eigenvalues, 325
  μ-synthesis method and, 336–337, 336*f*
  overview, 309
  process, 335–337
  robust control design using mixed sensitivity, 318–321, 320*f*
  standard configuration, 335, 336*f*
  total internal stability and design
    disturbance rejection requirements, 317–318, 318*f*
    loop shaping concept, 317–318, 318*f*
    mixed sensitivity approach, 318
    Nyquist diagram, 316*f*

robust stability, 315–316, 315*f*, 316*f*
total internal stability and design for disturbance rejection, 314–318
v gap metric: *see v* gap metric approach
see also Hardy spaces
'hand computation', 40
Hardy, G.H., 339
Hardy spaces, 310–312, 339
effect of choice of *p* on nature of norm (elementary illustration), 310, 311*f*
H_∞: *see* H_∞
non-elementary aside, 311–312, 312*f*
suffix *p*, 310
Hazen, H.L., 368
heating process, in feedback control, 22–23, 23*f*, 24*f*
Hilbert spaces, 372, 381
hill-climbing methods, 67–68, 67*f*
Hooke's law, 409, 410
Hopf, Heinz, 380
hot strip rolling mill (case study), 407–411, 408*f*–409*f*
thickness deviations, causes of, 410–411
transport lag problem, 407
human temperature regulation system, 2, 3*f*, 4
Hurwitz, A., 115–116
hydro frequency control, power amplification obtained in, 123, 123*f*
hysteresis, in non-linear systems, 95

IFAC: *see* International Federation of Automatic Control (IFAC)
imaginary axis, frequency response of system with poles and/or zeros and, 58–59, 59*f*
industrial application, automatic control schemes for, 177–182
methodologies and illustrations, 177–182, 178*f*–182*f*
inner loop
operational, jacketed reactor performance, 253–254, 254*f*
innovation sequence, 306
input shaping, 170–172, 171*f*
principle of, 170, 171*f*
instability theorem, 333
integrals
convergence, Laplace transform and, 30–31
convolution, 40–41, 40*f*
integral term
in removing any constant error, value of, 156
three term controller, 157, 157*f*
integrator, 48, 48*f*
intelligent and learning systems, literature resources, 378
intelligent machines, 343
intelligent systems, properties, 364–365, 364*f*
interdeterminism, in mathematical modelling, 78
interior point methods, 309
internal energy storage, system with, 6
International Date Line, thought experiments and, 82
International Federation of Automatic Control (IFAC), 199, 374
inverse function theorem, 107–108, 108*f*
inverse Laplace transformation, 29, 30, 124, 125, 160, 209
inverse Nyquist view, in *A* matrix of system representation, 254–256, 255*f*, 256*f*
inverse transfer function, open loop response of, 254–255
inverse Z transforms, non-uniqueness of, 197

investment, in automation: *see* industrial application, automatic control schemes for
investment strategy tool
  for n linked processes, 180, 180*f*
  for six processes with three choices at each stage, 180, 181*f*
irreversibility, in mathematical modelling, 78
iterative search in parameter space, visualisation of, 67, 67*f*

jacketed reactor
  cascade control and, 250–251, 250*f*, 251*f*
    inner loop into model, 253
    performance, 253–254, 254*f*
    state space equations by inspection from block diagram, 251–253
Jacobian matrix, 107
Jordan canonical form, 234–236
jump resonance, in non-linear systems, 95

Kalman, R.E., 247
Kalman filter
  application of, prerequisites for, 304, 304*f*
  complex processes and, 305
  continuous time processes and, 305
  disturbing signals and, 305
  idea, 306
  innovation sequence, 306
  Matlab demonstration of, 305–307, 307*f*
  non-Gaussian disturbance signals and, 305
  non-linear processes and, 305
  predictive control and, 307–308, 308*f*
  principle of, illustration of, 301*f*
  processes varying with time, 305
  state estimation, 300–302, 301*f*
  time-varying processes and, 305
Kalman gain, 302
  matrix, 302
Kalman's conjecture, in envelope methods, 96, 96*f*
kernel
  literature resources, 377
  loss of dimension into, 246

Laplace transform, 20, 29–45
  block manipulation and, 32–33, 33*f*
  closed loop system to desirable locations, 38–39, 38*f*, 39*f*
  complex plane, pole and zero locations in, 34–35, 34*f*, 35*f*, 36*f*
  in control theory, 31–32, 31*f*
  convergence of integral and, 30–31
  for convolution, 40–41, 40*f*
  damped natural frequency, 43–44, 44*f*
  definition of, 29–31
  for differential equation, 31, 33
  operation, 29*f*
  $0^-/0^+$ phenomenon and, 31
  poles and zeros in: *see* poles; zeros
  pole-zero diagrams: *see* pole-zero diagrams
  root locus technique: *see* root locus technique
  software packages and, 30
  techniques, 29
  time solutions calculation using transition matrix, 208, 209
  transfer function
    concept of, 32, 32*f*
    by cross-correlation, 41–42
    from differential equation, 33
    from frequency response curve, 39
    poles and zeros of, 33–34
    from transient response curve, 39, 40
  transform pairs and, tables of, 30

learning systems
with/without supervision, 361–364
abstract, structural characteristics of, 362–364, 362*f*, 363*f*
adaptivity *vs.*, 362
basic ideas, 361–362
Lebesgue, Henri, 337
Lebesgue spaces $L^p[a, b]$, 337–338
linear control systems, 25–26
operator theory and functional analysis applied to, 381
linear control theory, 102
linear finite-dimensional system, 204
linear infinite-dimensional system, 204
linearisation
comments, 105–106, 105*f*
about constant value, 105, 105*f*
about current solution, 105, 105*f*
definition of, 104–106
example, 104–105
of model, multivariable control theory, 205–206
motivation for, 103
about nominal solution, 105, 105*f*
about nominal trajectory, 106, 106*f*
linear matrix equality (LME), 332
Linear Matrix Inequalities (LMIs) method, 309
applications, 333
in control systems analysis and design, 330–334
defined, 332
discrete time linear system and, 333
feasibility stage, 332–333
linear system and, 333
overview, 330–332
robust pole placement using, 334, 335*f*
linear parameter varying (LPV)
gain scheduling
approach to, outline, 175
traditional ad hoc and, 173–175, 174*f*
linear quadratic Gaussian (LQG) approaches, 310
linear quadratic optimisation, 286–297
LQR problems with finite time horizon, 292–297, 292*t*, 293*f*–295*f*
LQR problems with infinite time horizon, 286–291
open loop stable process, 287–289, 288*f*
open loop unstable process, Riccati equation for, 289–291, 290*f*, 291*f*
Linear Quadratic Regulator (LQR) problems
with finite time horizon, 292–297, 292*t*, 293*f*–295*f*
with infinite time horizon, 286–291
open loop stable process, 287–289, 288*f*
open loop unstable process, Riccati equation for, 289–291, 290*f*, 291*f*
linear systems
characteristics, 2, 2*f*
frequency response of, 47–48, 48*f*
LMIs method and, 333
stability tests, 116, 116*t*
linear time invariant (LTI) controller, 335
LME: *see* linear matrix equality (LME)
LMI region, 334, 335*f*
LMIs: *see* Linear Matrix Inequalities (LMIs) method
local stability, 118, 118*f*
log-linear axes, for Bode diagrams, 49
loop shaping concept, 317–318, 318*f*
LPV: *see* linear parameter varying (LPV)
LQG: *see* linear quadratic Gaussian (LQG) approaches

LQR: *see* Linear Quadratic Regulator (LQR)
LTI controller: *see* linear time invariant (LTI) controller
Lyapunov, A.M., 333, 370
Lyapunov's first method, 95
Lyapunov's second (direct) method, 95
  geometric interpretation of, 120–121
  properties, 119
  for stability determination, 119–121, 119*f*–120*f*

machine intelligence, 343
Magellan, Ferdinand, 82
magnitude curve, 56, 57*f*
manually controlled process, 6, 6*f*
Maple, 374
mapping representation, of Σ, 203–204, 204*f*
Mathematica, 374
mathematical modelling, 63–91
  anticipatory systems, 72–74
  approaches to, 63–64, 63*f*
  causality, 76–77
  chaotic behaviour, 74–76, 75*f*
  complex physical systems, 77–78
  construction, from theoretical considerations, 66
  on data obtained by experimentation, 64–66
  determinism and predictability, 78
  development, methods for, 64
  difficulties, 68, 69*f*
    water wheels and, 79–82, 80*f*, 81*t*
  dimensional analysis, 85–86
  distributed systems in, 87–89
  doubt and certainty, 71–72
  experimentation on plants, 83–85, 83*f*, 84*f*, 85*f*
  literature resources, 379
  oscillating and osculating approximation of curves, 90–91, 91*f*
  parameter estimation, 69
    methods/approaches/techniques for, 66–68, 67*f*
  parameters, fixing of, 69
  point representation, spatial region and, 89–90, 90*f*
  regression analysis, 69–70, 70*f*
  residuals, analysis of, 70–71, 71*f*
  reversibility and irreversibility, 78
  social and political phenomena, 79
  thought experiments, effectiveness of: *see* thought experiments
  time-ordering, 77
Mathworks, 374
Matlab, 335, 374
  commands for RGA, 268
  demonstration, of Kalman filter, 305–307, 307*f*
  μ-synthesis programme, 336–337
    D-K iteration, 337
matrix exponential, time solutions calculation using transition matrix, 207
Maxwell, J. C., 77–78
'method of coefficients,' 81
Mobile Reference, 30
modal analysis, 256–264, 257*f*, 258*f*, 259*f*, 260*f*, 261*f*
  for third order system with two complex eigenvalues, 261–264
modal control, by state feedback, 239–245
modal matrix, 208
model-generated data, recorded system *vs.*, 67
Model Predictive Control (MPC), 308, 308*f*
modes, 256–264, 257*f*, 258*f*, 259*f*, 260*f*, 261*f*
MPC: *see* Model Predictive Control (MPC)
multi-agents, 378
multilayer networks, 347, 348*f*

multiplicative uncertainty model, 322*f*
multivariable linear systems and state space approach
  alternative system descriptions, 202–203, 202*f*, 202*t*
  canonical representations, 224–236, 224*f*, 225*f*
    advantages, 224–226
    controllability canonical form (phase-variable form), 228–231, 228*f*
    Jordan canonical form, 234–236
    observability canonical form, 231–232, 232*f*
    reachability canonical form, 226–228, 226*f*
    reconstructibility canonical form, 232, 233*f*
    state equations for multi-input, multi-output processes, 233–234
  continuous time systems: *see* continuous time systems
  control system design, 236–237
    state-variable feedback, 237–239
  discrete time state equation
    with forcing, 221
    transition matrix, 220
    unforced, 219–220
  discrete time system: *see* discrete time system
  equivalent systems, 216
  linear transformations, conservation of dimension under, 245–247, 246*f*
  mapping representation of Σ, 203–204, 204*f*
  modal control by state feedback, 239–245
  realisation, 216–217
    existence, 217
    uniqueness, 217
  representations, 199
  stability, 217–218
    tests, 222
  state, concept of, 201
  state vector, 199–201, 200*f*
  transfer-matrix and vector–matrix description, 215
  transition matrix: *see* transition matrix
multivariable processes, control of
  RGA in: *see* relative gain array (RGA)
  singular value decomposition, 268–269
μ-synthesis method, 309
  H_∞ and, 336–337, 336*f*
mutation mechanism, 356, 357*f*

natural frequency, of system, 109–110
neural networks, 343
  architectures, for dynamic processes, 349–352, 349*f*, 350*f*, 351*f*
  literature resources, 377
  training, 347–349
  see also artificial neural networks (ANN)
neurons, in ANN
  architecture of, 345, 345*f*
  properties, demonstration of, 345–347, 346*f*
    with *n* inputs, 347
neutral point (NP), of aircraft, 133, 134*f*
Nichols chart, 60–61
Nikon Speedlight, 74
nominal trajectory, linearisation about, 106, 106*f*
non-Gaussian disturbance signals, Kalman filter and, 305
non-linear controllers, 22
non-linear processes, Kalman filter and, 305
non-linear systems, 93–110
  approaches to analysis of, 95–96

chaotic behaviour (example), 109–110, 110*f*
classification of systems in, 94*f*
control loops, analysis of, 97–98, 97*f*
describing function method: *see* describing function method
forced pendulum (example), 109–110, 110*f*
jump resonance, stick-slip motion, backlash and hysteresis in, 95
linear approximation, derivative as, 107, 107*f*
inverse function theorem, 107–108, 108*f*
linearisation: *see* linearisation
Lyapunov's first method, 95
Lyapunov's second/direct method, 95
overview, 93–95
process, 101, 102–103, 103*f*
relay and switched systems in, 95
second-order linear systems in state plane, 99–100, 99*f*, 100*f*
sector bound methods, 96, 96*f*
signal problems
large, 101, 102
small, 102–103, 103*f*
stability tests, 117, 117*f*
stable oscillators in, 95
in state plane
second-order linear systems, 99–100, 99*f*, 100*f*
second-order systems in, 100–101, 102*f*
transversality, concept of, 108, 109*f*
non-minimum phase systems, 39, 135–139, 136*f*, 141–142, 142*f*, 144
physical phenomena, 136
pole–zero diagram, 135, 136*f*
non-periodic behaviour, 76
norm, scalar-valued cost function, 276
NP: *see* neutral point (NP)
Nyquist, Harold, 47
Nyquist diagram, 60–61
Nyquist stability criterion
unstable process, 130–133, 131f, 132f

observability, 247–248
continuous time systems, 218
discrete time system, 222–224
observability canonical form, 231–232, 232*f*
observability Gramian, 248
open loop control problems, 236
open loop response, of inverse transfer function, 254–255
operator theory, literature resources, 381
optimal control problem, 236
calculus of variations and, 275–276
optimisation, 271–297
linear quadratic optimisation, 286–297
see also linear quadratic optimisation
literature resources, 376
maximum value
mathematical function and, 271, 272*f*
mathematical function on restricted interval and, 272, 272*f*
scalar-valued cost function, 273–275, 274*f*, 275*f*, 276–277, 276*f*–277*f*
scalar-valued function of *n* variables, 272–273, 273*f*
scalar-valued function on closed region of parameter space, 273, 274*f*
overview, 271
time-optimal control, 278–286, 279*f*–280*f*
see also time-optimal control
ordinary differential equations, 379–380

oscillating approximation, of curves, 90–91, 91*f*
osculating approximation, of curves, 90–91, 91*f*
overfitting, 348–349, 349*f*
overtraining, 348–349, 349*f*
oxygen steelmaking: *see* Basic Oxygen Steelmaking (BOS) process

pairing, of actuators, 268
parameters, in mathematical modelling
  estimation, 69
    methods/approaches/techniques for, 66–68, 67*f*
  fixing, 69
PCA: *see* principal components analyser (PCA)
Peano–Baker series, 213
perturbation
  stable system and, 111–112, 112*f*
  unstable system and, 111–112, 112*f*
phenomenological laws, 77
PLCs: *see* programmable logic controllers (PLCs)
poles
  cancellation, 37
  of closed loop system to desirable locations, 38–39, 38*f*, 39*f*
  frequency response of system with, 58–59, 59*f*
  locations, 176
  locations, in complex plane, 34–35, 34*f*, 36*f*
    meaning of, 34–35, 35*f*
  open loop, zeros and
    root locus of system, 45, 45*f*
  placement, 35, 37–38, 37*f*
  pole-zero diagrams: *see* pole-zero diagrams
  of state space systems, 265–266
  system
    presumed initial position of, 37*f*
    required position of, 37*f*
  of transfer function, 33–34
pole-zero diagrams, 37*f*
  associated system step responses and, 36*f*
  frequency response and, imaginary axis, 58–59, 59*f*
  resonant frequencies from, 42–43, 43*f*
  of second-order system, 45, 45*f*
Pontryagin maximum principle, 283
position control, by state space methods, 249
power amplification, in hydro frequency control, 123, 123*f*
predictability, in mathematical modelling, 78
pre-programmed control, feedback control and, 28
pressure control, inside small fuel-fired furnaces, 414–417, 415*f*, 416*f*
principal components analyser (PCA), 352
probabilistic determinism, in mathematical modelling, 78
probing controller, 169–170, 169*f*–170*f*, 365–366
programmable logic controllers (PLCs), 185
proportional term, three term controller, 157, 157*f*
pupillary servomechanism, 73
quadratic behaviour, 194–196, 195*f*, 196*f*

reachability
  continuous time systems, 218
  discrete time system, 222–224
reachability canonical form, 226–228, 226*f*
real plane, two-dimensional
  simple properties of neuron in, 345–347, 346*f*

real systems, 65
real-world systems, non-linearity in, 94, 94*f*
reconstructibility
  continuous time systems, 218
  discrete time system, 222–224
recorded system data, model-generated *vs.*, 67
regression analysis, in mathematical modelling, 69–70, 70*f*
regulator problem, defined, 236
relative gain, 266
relative gain array (RGA), 266–269
  forecaster of steady state interaction, 266–267
  illustrative example, 267–268
  Matlab commands for, 268
relay controller, 149–154
  advantages, 151
  disadvantages, 151
  rudimentary on–off control, 149–151, 149f
  sliding mode control, 151–155, 152*f*, 153*f*, 154*f*
  variable structure control, 151–155
relay control systems, 150, 150*f*, 151
relay systems, in non-linear systems, 95
'requirement capture,' 16, 17*f*
residuals, analysis of, 70–71, 71*f*
resonant frequencies, from pole-zero diagrams, 42–43, 43*f*
reverse engineering, 16, 17*f*
reversibility, in mathematical modelling, 78
Reynolds number, 86
RGA: *see* relative gain array (RGA)
Riccati, Count Jacopo, 285
Riccati equation, 285–286
  open loop unstable process, 289–291, 290*f*, 291*f*
Riesz, F., 339
robust control design
  adaptivity and, 337
  ΔG envelope, specification of, 322
  H_∞: *see* H_∞
  Hardy spaces, 310–312
  literature resources, 376–377
  LMI method: *see* Linear Matrix Inequalities (LMIs) method
  overview, 309–310
  spaces, hierarchy of, 337–339
'Robust/Fragility trade-off,' 319
robustness against changes
  control loop, 124–125, 125*f*
robust stability, of closed loop system, 313–314, 314*f*
root locus technique, 38–39, 39*f*
  with open loop poles and zeros, 45, 45*f*
Rosen, R., 72–73
Routh, E.J., 116, 370
Routh–Hurwitz criterion, 116
rudimentary on–off control, 149–151, 149f
rule of thumb, for control
  fuzzy control algorithm and, 354–355, 355*f*
Runge–Kutta approach, 106

Saturn's rings, stability studies, 112, 113*f*
scalar multiplier, 93
scalar-valued coefficients, 89
Scilab, 253, 335, 374
second-order systems
  frequency response of, 57–58, 58*f*
  linear, in state plane, 99–100, 99*f*, 100*f*
  non-linear, in state plane, 100–101, 102*f*
sector bound methods, 96, 96*f*
self-organising maps (SOM)
  neural net-based, for data reduction and clustering, 352
self-tuning controllers
  adaptivity and, 361
  into closed loop, 165
  coefficients, three-term controller, 165

sensitivity functions, limitations on performance, 141, 143
sensor systems, 419
separation principle, 299–300
sequence spaces l^p, 338
series summation method
  time solutions calculation using transition matrix, 207, 210, 210*t*–211*t*
servomechanism, pupillary, 73
Shannon's theorem, 141, 372*t*
Σ, mapping representation of
  multivariable linear systems and state space approach, 203–204, 204*f*
signal non-linearity
  large, 101, 102
  small, 102–103, 103*f*
Simulink, 374
single-input, single-output (SISO) systems, 250
  control loops
    RGA: *see* relative gain array (RGA)
    singular value decomposition, 268–269
  state space model for, 255
singular value decomposition (SVD), 268–269
singular values
  eigenvalues and, 325
  H_∞ norms from dynamic process models, 322–324
sliding mode control, 151–155, 152*f*, 153*f*, 154*f*, 376
  literature resources, 376
sliding motion, 152, 153*f*
software
  control-oriented, 374
  engineering context, 14, 16–17, 16*f*, 17*f*
SOM: *see* self-organising maps (SOM)
spaces
  Hardy spaces: *see* Hardy spaces
  hierarchy of, 337–339, 338*f*
  inclusion relations between, 338–339
  Lebesgue spaces $L^p[a, b]$, 337–338
  norm of linear mapping T, 339
  sequence spaces l^p, 338
  see also H_∞
spatial region representation, as summation of elemental regions, 88
speed of response, 176
  control designer and, 176
stability, 111–113, 111*f*–112*f*, 198
  of closed loop system, 49, 49*f*
  determination, Lyapunov's second (direct) method, 119–121, 119*f*–120*f*
  frequency response and, 48–49, 49*f*
  local and global, 118, 118*f*
  margins, 116
    frequency response and, 50
  quantification, 113–116, 114*t*
  Saturn's rings, 112, 113*f*
  tests
    continuous time systems, 217–218
    discrete time system, 222
    linear system, 116, 116*t*
    non-linear systems, 117, 117*f*
  theory, history, 112–113, 113*f*
stable oscillators, in non-linear systems, 95
stable system, 111, 111*f*
  perturbation and, 111–112, 112*f*
state
  concept of, 201
  parameter estimator and, 237
state equation(s)
  discrete time: *see* discrete time state equation
  £ transform equivalent of, 221–222
  for multi-input, multi-output processes, 233–234
state estimation, 299–308
  application, 300, 300*f*

Kalman filter, 300–302, 301*f*
see also Kalman filter
literature resources, 376
optimal gain matrix and, 303–304, 304*f*
overview, 299
separation principle, 299–300
state estimator, 237
state feedback, modal control by, 239–245
state observer, 237
state plane
linear second-order systems in, 99–100, 99*f*, 100*f*
non-linear second-order systems in, 100–101, 102*f*
state portrait, 99
state space, 201, 249–269
inverse Nyquist approach, 254–256, 255*f*, 256*f*
modal analysis of, 249
modes and modal analysis, 256–264, 257*f*, 258*f*, 259*f*, 260*f*, 261*f*
multivariable linear systems and: *see* multivariable linear systems and state space approach
overview, 249–250
poles and zeros and, 265–266
position control by, 249
RGA: *see* relative gain array (RGA)
singular value decomposition, 268–269
SISO and, 250, 255
transfer function and, 264–266
view of cascade control, 249, 250–254, 250*f*
state-variable feedback, control system design and, 237–239
state-variable feedback strategy, 237
state vector, 199–201, 200*f*
step response of system
coefficients, three-term controller, 158–164, 159*f*, 161*f*–163*f*
order of process and, 163–164, 164*f*
system zeros on, 136–139, 137*f*, 139*f*
stick-slip motion, 102–103
in non-linear systems, 95
Stodola, A.B., 115
strong determinism, in mathematical modelling, 78
support vector machines (SVM), 352
support vector methods, literature resources, 377
SVD: *see* singular value decomposition (SVD)
SVM: *see* support vector machines (SVM)
swarm intelligence, 378
switched systems
disadvantages of, 150
in non-linear systems, 95
switching line, 151, 153*f*
switching surface, 151
synthesis, limits on, 126
system bandwidth, 176
control designer and, 176
system natural frequency, 176
systems
boundary, 1
defined, 1, 3, 4
dynamic, 2, 6
feedback, 2, 3*f*
general ideas, 1–2
human temperature regulation, 2, 3*f*, 4
linear, 2, 2*f*
realisation, 216–217
existence, 217
uniqueness, 217
structure of, 1, 1*f*
zeros, on step response of system, 136–139, 137*f*, 139*f*

Takahashi's algorithm, 191–192, 192*f*
telephony, 165–166, 166*f*
feedback amplifiers and, 165–166, 166*f*

theory of equations, 381
third order system, two complex eigenvalues and
  modal analysis for, 261–264
thought experiments
  about water wheels, 82
  effectiveness of, 82
  International Date Line, conjecturing about, 82
three-term controller, 154–166
  coefficients, 158–165, 158*f*, 161*f*–162*f*, 163*f*
    controller gain and, 164–165
    self-tuning algorithm, 165
    step response, 158–164, 159*f*, 161*f*–163*f*
  concept of, 155–156
  derivative term, 157, 157*f*
    to control degree of damping, value of, 156–157, 156*f*, 157*f*
  idea of, 155–156
  integral term, 157, 157*f*
    in removing any constant error, value of, 156
  proportional term, 157, 157*f*
time domains
  frequency and, 325
  integral constraints, limitation on performance and, 143–144, 144*f*
time-invariant systems
  continuous time, controllability and observability tests for
    by diagonalisation method, 219
    by matrices *Qc, Qo,* 218
  time solutions calculation using transition matrixs, 207–211
time-optimal control, 278–286, 279*f*–280*f*
  construction of, 284–285
  geometric view, 280–283, 281*f*–283*f*
time-ordering, in mathematical modelling, 77
time-varying processes
  Kalman filter and, 305
  time solutions calculation using transition matrix, 211–213
topology
  concept of, 2
  differential, 380
Torricelli, theorem of, 80
tracking problem, defined, 236
trajectories
  crossing of, 110
  defined, 99
transfer function
  concept of, 32, 32*f*
  by cross-correlation, 41–42
  from differential equation, 33
  from frequency response curve, 39
  poles and zeros of, 33–34
  state space and, 264–266
    poles and zeros and, 265–266
  of system, frequency response testing and, 56–57, 57*f*
  from transient response curve, 39, 40
transfer-matrix, vector–matrix description and, 215
transient response, of three-tank system, 258–259, 258*f*
transient response curve, transfer function from, 39, 40
transition matrix
  defined, 212
  of discrete time equation, 220
  properties of, 212, 215
  time solutions calculation using
    diagonalisation approach, 208–211
    fundamental matrix, 212
    Laplace transform approach, 208, 209
    matrix exponential, 207
    periodically time-varying case, 214
    series summation method, 207, 210, 210*t*–211*t*

time-invariant case, 207–211
time-varying case, 211–213
transport lag problem, 407
trans-US telephone cable, 166*f*
transversality, concept of, 108, 109*f*
tuning procedure, on-line, 164–165

unforced discrete time state equation, 219–220
existence and uniqueness of solution, 219–220
unit step response of system, 53–54, 53*f*
universal constants, 66
universality of control theory, 3
unstable pole
cancellation of, by matching zero the controller, 127–128, 128*f*
shifting, by feedback, 128–129, 128*f*
unstable system, 111, 111*f*
control, 126–135
cancellation of unstable pole by matching zero, 127–128, 128*f*
shifting unstable pole by feedback, 128–129, 128*f*
design, 133–135, 134*f*
Nyquist stability criterion, 130–133, 131f, 132f
perturbation and, 111–112, 112*f*

van der Pol equation, 95
variables, control design, 149
variable structure control, 154
vector–matrix
transfer-matrix description and, 215
velocity profile, 278, 279*f*
vertical standing in humans
control of, 129–130, 130*f*
V function contour, 119–120, 119*f*, 120*f*
v gap metric approach, 309, 326–330
$\delta\nu$ and $b_{G,D}$, 327–328, 327*f*, 329–330
insights, 328, 329*f*
overview, 326
use of, 326–327
Vissim, 374

water wheels
mathematical modelling difficulties and, 79–82, 80*f*, 81*t*
thought experiment about, 82
Watt, James, 368
weak determinism, in mathematical modelling, 78
Weierstrass' theorem, 272, 352
Wigner, Eugene, 77

Xcos, 374
XOR function, 347

zeros
frequency response of system with, 58–59, 59*f*
locations, in complex plane, 34–35, 34*f*
open loop poles and, root locus of system, 45, 45*f*
pole–zero diagrams: *see* pole-zero diagrams
of state space systems, 265–266
on system response, 36*f*
of transfer function, 33–34
Ziegler–Nichols
equations, coefficients of three-term controller and, 158–159, 164
methods, 54
Z transforms, non-uniqueness of inverse, 197, 197*f*

Control Theory

A guided tour

3rd Edition

Using clear tutorial examples, this fully updated new edition concentrates on explaining and illustrating the concepts that are at the heart of control theory.

It seeks to develop a robust understanding of the underlying principles around which the control subject is built. This simple framework is studded with references to more detailed treatments and also has interludes that are intended to inform and entertain.

The book is intended as a companion on the journey through control theory and although the early chapters concentrate on fundamental ideas such as feedback and stability, later chapters deal with more advanced topics such as state variables, optimisation, estimation, Kalman filtering and robust control.

Professor James Ron Leigh (C Eng., C Math., FIET, FIMA) studied Electrical Engineering at Sheffield College of Technology, winning the Freshgate Scholarship. He has an MSc in Maths and a PhD in Control Engineering. He worked as a Systems Engineer at United Steel Companies and as Head of Control Systems Development at British Steels Corporate Labs. As Post-Above-Reader at the University of Westminster, he set up university-wide interdisciplinary research projects and was Founder/Director of the Industrial Control Centre and the Government/Industry funded Advanced Control Club. His specialisation continues to be assisting the transfer of useful technology from academia to industry. He has served the IET in several capacities, particularly in relation to control applied to biotechnology.

ISBN 978-1-84919-227-9

9 781849 192279

The Institution of Engineering and Technology
www.theiet.org
978-1-84919-227-9