Lösningar Linjära system okt 2014

1a. $F_s = 25\,kHz \Rightarrow t = n \cdot T_s = \dfrac{n}{25000} \Rightarrow$

$$x[n] = 10^6 \cdot \frac{n}{25000} \cdot e^{-\frac{2500}{25000} \cdot n} \cdot \cos \frac{12500\pi}{25000} \cdot n =$$

$$= \underline{\underline{40 \cdot n \cdot 0{,}6065^n \cdot \cos[n \cdot 0{,}5\pi]}}$$

1b. $$f(t) = \frac{3e^t + 2t}{e^{5t}} = 3 \cdot e^{-4t} + 2t \cdot e^{-5t} \supset \underline{\underline{\frac{3}{s+4} + \frac{2}{(s+5)^2}}}$$

1c. $$X(z) = \frac{2}{4 + \frac{1}{z}} = \frac{2z}{4z+1} = \frac{1}{2} \cdot \frac{z}{z+0{,}25} \subset \underline{\underline{\frac{1}{2} \cdot (-0{,}25)^n}}$$

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2. $$H(z) = \frac{Y(z)}{X(z)} = \frac{z}{z-0{,}6} \qquad x[n] = \sigma[n] \Rightarrow$$

$$X(z) = \frac{z}{z-1} \Rightarrow Y(z) = \frac{z}{z-1} \cdot \frac{z}{z-0{,}6} = z \cdot \frac{z}{(z-1)(z-0{,}6)}.$$

$$\Rightarrow N'(z) = 1 \cdot (z-0{,}6) + (z-1) \cdot 1.$$

Poler: $z = \begin{cases} 1 \\ 0{,}6 \end{cases} \Rightarrow y[n] = A + B \cdot 0{,}6^n$

$$A = \left[\frac{T}{N'}\right]_{z=1} = \frac{1}{1-0{,}6} = 2{,}5 \qquad B = \left[\frac{T}{N'}\right]_{z=0{,}6} = \frac{0{,}6}{0{,}6-1} = -1{,}5$$

$$\Rightarrow \underline{\underline{y[n] = 2{,}5 - 1{,}5 \cdot 0{,}6^n}}$$

3a. poler: $z^2 + 0{,}8z + a = 0 \Rightarrow$

$$z = -0{,}4 \pm \sqrt{0{,}16 - a}$$

Vid reella poler skall $\sqrt{0{,}16-a} > 0{,}6 \Rightarrow$

$0{,}16 - a > 0{,}36 \Rightarrow a < -0{,}2$.

Tag t.ex. $\underline{\underline{a = -0{,}5}}$

3b. $$H(z) = \frac{z^2}{z^2+0{,}8z-0{,}2} = z \cdot \frac{z}{z^2+0{,}8z-0{,}2} = z \cdot \frac{T}{N}$$

$N' = 2z + 0{,}8$.

Poler: $z^2+0{,}8z-0{,}2=0 \Rightarrow z = -0{,}4 \pm \sqrt{0{,}16+0{,}2} = \begin{cases} -1 \\ 0{,}2 \end{cases}$

$$\Rightarrow \quad h[n] = A\cdot(-1)^n + B\cdot 0{,}2$$

$$A = \left[\frac{T}{N'}\right]_{z=-1} = \frac{-1}{-2+0{,}8} = \frac{-1}{-1{,}2} = \frac{5}{6}$$

$$B = \left[\frac{T}{N'}\right]_{z=0{,}2} = \frac{0{,}2}{0{,}4+0{,}8} = \frac{0{,}2}{1{,}2} = \frac{1}{6} \Rightarrow$$

$$\underline{\underline{h[n] = \frac{5}{6}\cdot(-1)^n + \frac{1}{6}\cdot 0{,}2^n}}$$

4. $y[n] = a \cdot x[n] + x[n-1] + by[n-1] + c \cdot y[n-2] \Rightarrow$

$$Y(z) = aX(z) + z^{-1}X(z) + bz^{-1}Y(z) + c \cdot z^{-2}Y(z) \Rightarrow$$

$$Y(z)\left(1 - bz^{-1} - c \cdot z^{-2}\right) = X(z)\left(a + z^{-1}\right) \Rightarrow$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{a + z^{-1}}{1 - bz^{-1} - cz^{-2}} = \underline{\underline{\frac{az^2 + z}{z^2 - bz - c}}}$$

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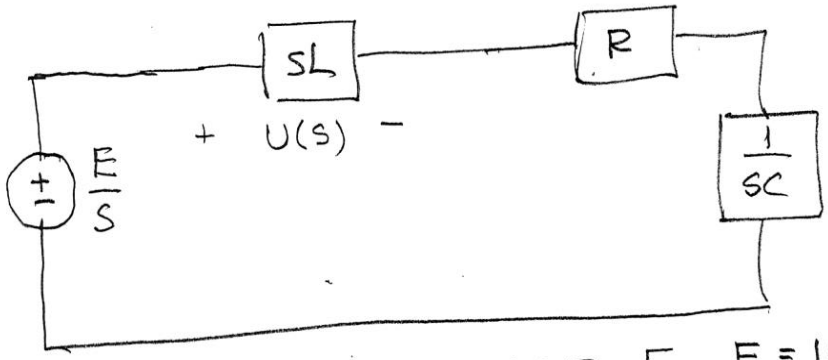
5. $x(t) = 4 - 7\sin 12000t + 12\sin(8800\pi t - 30^\circ) - 8 \cdot \cos 1840\pi t$

$$\Rightarrow \omega_{max} = 8800\pi \approx 27646 \text{ rad/s} \Rightarrow$$

$$f_{max} = 4400 \text{ Hz} \Rightarrow$$

$$f_s > 2 \cdot f_{max} = \underline{\underline{8800 \text{ Hz}}}$$

6a. Laplace-schema, $t>0$



$R = 100\,\Omega \quad L = 1\,H \quad C = 625\,\mu F \quad E = 10V$

Spänningsdelning ger

$$U(s) = \frac{sL}{sL + R + \frac{1}{sC}} \cdot \frac{E}{s} = \frac{sLC}{s^2LC + sRC + 1} \cdot E =$$

$$= \frac{s \cdot E}{s^2 + \frac{R}{L} \cdot s + \frac{1}{LC}} = \frac{s \cdot 10}{s^2 + 100s + 1600} = \frac{T}{N}$$

$N' = 2s + 100$

Poler: $s^2 + 100s + 1600 = 0 \Rightarrow s = \begin{cases} -80 \\ -20 \end{cases} \Rightarrow$

$$u(t) = A \cdot e^{-80t} + B \cdot e^{-20t}$$

$$\left. \begin{aligned} A &= \left[\frac{T}{N'}\right]_{s=-80} = \frac{-80 \cdot 10}{-160 + 100} = \frac{40}{3} \approx 13{,}3 \\ B &= \left[\frac{T}{N'}\right]_{s=-20} = \frac{-20 \cdot 10}{-40 + 100} = -\frac{10}{3} \approx -3{,}33 \end{aligned} \right\} \Rightarrow$$

$$\underline{\underline{u(t) = \frac{40}{3} \cdot e^{-80t} - \frac{10}{3} \cdot e^{-20t}}} \approx \underline{\underline{13{,}3e^{-80t} - 3{,}33 \cdot e^{-20t}}}$$

6b. $u(t)$ svängande $\Rightarrow$ komplexa poler

$$U(s) = \frac{s \cdot E}{s^2 + \frac{R}{L} \cdot s + \frac{1}{LC}} = \frac{s \cdot E}{s^2 + 100s + \frac{1}{C}}$$

Poler: $s^2 + 100s + \frac{1}{C} = 0 \Rightarrow$

$$s = -50 \pm \sqrt{2500 - \frac{1}{C}}$$

Komplexa poler $\Rightarrow$ $2500 - \frac{1}{C} < 0 \Rightarrow$

$$C < \frac{1}{2500} F = \underline{\underline{400 \mu F}}$$

7. $T = 1{,}5 - (-0{,}5) = 2 \Rightarrow \underline{\underline{\omega_0}} = \frac{2\pi}{T} = \underline{\underline{\pi}}$

$\underline{k=0}$

$$c_0 = \frac{1}{T}\int_{1\,\text{period}} x(t)\,dt = \frac{1}{2}\cdot\int_{-0{,}5}^{0{,}5} 2\,dt = 1 \Rightarrow \underline{\underline{a_0 = 1}}$$

$\underline{k \neq 0}$

$$c_k = \frac{1}{T}\int_{1\,\text{period}} x(t)\,e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_{-0{,}5}^{0{,}5} 2\,e^{-jk\omega_0 t}\,dt =$$

$$= \frac{1}{T}\left[\frac{2}{-jk\omega_0}\cdot e^{-jk\omega_0 t}\right]_{-0{,}5}^{0{,}5} = \frac{2}{-jk\omega_0 T}\cdot\left(e^{-jk0{,}5\omega_0} - e^{jk0{,}5\omega_0}\right) =$$

$$= \frac{2}{-jk2\pi}\cdot\left(-2j\sin 0{,}5k\omega_0\right) = \underline{\underline{\frac{2}{k\pi}\sin k\cdot\frac{\pi}{2}}}$$

$$a_k = 2\,\mathrm{Re}\,c_k = \frac{4}{k\pi}\sin k\frac{\pi}{2} \Rightarrow \underline{\underline{a = \frac{4}{\pi} \approx 1{,}27}}$$

$$b_k = -2\,\mathrm{Im}\,c_k = 0 \Rightarrow \qquad \underline{\underline{a_2 = \frac{4}{2\pi}\sin\pi = 0}}$$

$\underline{\underline{b_1 = 0}}$ $\qquad a_3 = \frac{4}{3\pi}\sin\frac{3\pi}{2} = \underline{\underline{-0{,}424}}$

$\underline{\underline{b_2 = 0}}$

$\underline{\underline{b_3 = 0}}$

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8. nollstället ges av $\Omega_0 = 2\pi \cdot \frac{f_0}{f_s} = 2\pi \cdot \frac{50}{300} = \frac{\pi}{3}$ rad $= 60°$

$$\Rightarrow z_{noll} = e^{\pm j\Omega_0} = e^{\pm j60°} = 0,5 \pm j0,5\sqrt{3} \Rightarrow$$

$$z_{pol} = 0,9 \cdot z_{noll} = 0,45 \pm j0,45\sqrt{3} \Rightarrow$$

$$H(z) = K \cdot \frac{(z-0,5-j0,5\sqrt{3})(z-0,5+j0,5\sqrt{3})}{(z-0,45-j0,45\sqrt{3})(z-0,45+j0,45\sqrt{3})} =$$

$$= K \cdot \frac{(z-0,5)^2+(0,5\sqrt{3})^2}{(z-0,45)^2+(0,45\sqrt{3})^2} = K \cdot \frac{z^2-z+1}{z^2-0,9z+0,81}$$

$$H(1) = 1 \Rightarrow K \cdot \frac{1-1+1}{1-0,9+0,81} \Rightarrow K = 0,91$$

$$\Rightarrow H(z) = 0,91 \frac{z^2-z+1}{z^2-0,9z+0,81} =$$

$$= 0,91 \cdot \frac{1-z^{-1}+z^{-2}}{1-0,9z^{-1}+0,81z^{-2}} = \frac{Y(z)}{X(z)} \Rightarrow$$

$$\underline{\underline{y[n] = 0,91x[n] - 0,91x[n-1] + 0,91x[n-2] + 0,9y[n-1] - 0,81y[n-2]}}$$

9. s-planet:

$$Y(s) = H_1(s)\cdot\left(X(s) - H_2(s)\cdot Y(s)\right) \Rightarrow$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)\cdot H_2(s)} =$$

$$= \frac{\frac{5}{s+5} - \frac{4}{s+4}}{1 + \left(\frac{5}{s+5} - \frac{4}{s+4}\right)\cdot\frac{A}{s}} = \frac{\frac{s}{(s+5)(s+4)}}{1 + \frac{s}{(s+5)(s+4)}\cdot\frac{A}{s}} =$$

$$= \frac{s}{(s+5)(s+4) + A} = \frac{s}{s^2+9s+20+A}$$

poler: $s^2+9s+20+A = 0$

$$s = -4{,}5 \pm \sqrt{20{,}25-20-A} = -4{,}5 \pm \sqrt{0{,}25-A}$$

Reella poler: Stabilt om $\sqrt{0{,}25-A} < 4{,}5 \Rightarrow$

$$0{,}25 - A < 20{,}25 \Rightarrow \underline{A > -20}$$

Komplexa poler: $A > 0{,}25$

$\Rightarrow$ Stabilt om $\underline{\underline{A > -20}}$

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10. Differensekvation

$y[n] = a \cdot x[n] + b y[n-1] \Rightarrow$

$Y(z) = a X(z) + b z^{-1} Y(z) \Rightarrow$ överföringsfkn

$$H(z) = \frac{Y(z)}{X(z)} = a \cdot \frac{1}{1 - b z^{-1}} = a \cdot \frac{z}{z-b}$$

Frekvensfunktion

$$H(e^{j\Omega}) = a \cdot \frac{e^{j\Omega}}{e^{j\Omega} - b} = a \cdot \frac{e^{j\Omega}}{\cos\Omega - b + j\sin\Omega}$$

Amplitudfunktion

$$|H(e^{j\Omega})| = |a| \cdot \frac{|e^{j\Omega}|}{\sqrt{(\cos\Omega - b)^2 + \sin^2\Omega}} =$$

$$= |a| \cdot \frac{1}{\sqrt{\cos^2\Omega - 2b\cos\Omega + b^2 + \sin^2\Omega}} =$$

$$= |a| \cdot \frac{1}{\sqrt{1 + b^2 - 2b\cos\Omega}}.$$

$H(e^{j\Omega})$ är avtagande i $0 \leq \Omega < \pi$

Normerad gränsvinkelfrekvens Ω_g:

$$\Omega_g = 2\pi \cdot \frac{f_ö}{f_s} = 2\pi \cdot \frac{150\,Hz}{2000\,Hz} = 0{,}15\pi \text{ rad} = 27^0$$

(se diagram)

$$\Rightarrow \left|H(e^{j0,15\pi})\right| = \frac{1}{\sqrt{2}}\left|H(0)\right| \Rightarrow$$

$$\frac{|a|}{\sqrt{1+b^2-2b\cos 27^\circ}} = \frac{1}{\sqrt{2}} \cdot \frac{|a|}{\sqrt{1+b^2-2b\cos 0^\circ}}$$

$$\Leftrightarrow$$

$$1+b^2-2b\cos 27^\circ = 2(1+b^2-2b) \Rightarrow$$

$$b^2-(4-2\cos 27^\circ)b+1=0$$

$$b^2-2{,}218b+1=0 \Rightarrow$$

$$b = \begin{cases} 1{,}588 & (\text{ej stabilt}) \\ 0{,}630 \end{cases} \Rightarrow \underline{\underline{b=0{,}630}}$$

$$\left|H(e^{j0})\right| = 1 \quad (\text{diagram}) \Rightarrow$$

$$\frac{|a|}{\sqrt{1+b^2-2b}} = 1 \Rightarrow |a| = 0{,}37 \Rightarrow$$

$$\underline{\underline{a = \pm 0{,}37}}$$