Lösningar Linjära system okt 2011

1a. $T_s = \frac{1}{25k} = 4\cdot 10^{-5}$ ms $\Rightarrow t = 4\cdot 10^{-5}\cdot n$

$$x[n] = 10^5\cdot 4\cdot 10^{-5}\cdot n\cdot e^{-1600\cdot 4\cdot 10^{-5}n}\cdot \cos[10540\pi\cdot 4\cdot 10^{-5}\cdot n] =$$

$$= 4\cdot 0{,}9380^n\cdot \cos[0{,}4216\pi\cdot n]$$

1b. $f(t) = \frac{3+2e^t}{e^{4t}} = 3\cdot e^{-4t} + 2e^{-3t} \supset \frac{3}{s+4} + \frac{2}{s+3}$

1c. $X(z) = \frac{2}{0{,}2z^{-1}+1} = \frac{2z}{z+0{,}2} \Rightarrow x[n] = 2\cdot(-0{,}2)^n$

---

2. $h[n] = (-0{,}4)^n \Rightarrow H(z) = \frac{z}{z+0{,}4}$.

$x[n] = \sigma[n] \Rightarrow X(z) = \frac{z}{z-1} \Rightarrow$

$$Y(z) = H(z)\cdot X(z) = \frac{z^2}{(z+0{,}4)(z-1)} = z\cdot\frac{z}{(z+0{,}4)(z-1)} =$$

$= z\cdot \frac{T}{N}$; $N' = 1\cdot(z-1)+(z+0{,}4)\cdot 1$

Poler : $z = \begin{cases} -0{,}4 \\ 1 \end{cases} \Rightarrow y[n] = A\cdot(-0{,}4)^n + B\cdot 1^n$

$$A = \left[\frac{T}{N'}\right]_{z=-0{,}4} = \frac{-0{,}4}{-1-0{,}4} = \frac{2}{7}$$

$$B = \left[\frac{T}{N'}\right]_{z=1} = \frac{1}{1+0{,}4} = \frac{5}{7} \Rightarrow$$

$$y[n] = \frac{2}{7}\cdot(-0{,}4)^n + \frac{5}{7}$$

3a) Poler: $z = -0{,}4 \pm \sqrt{0{,}16 - a}$. Instabilt om $|z| > 1$.

Rella poler ($a \leq 0{,}16$) $\sqrt{0{,}16-a} + 0{,}4 > 1 \Rightarrow \underline{a < -0{,}2}$

komplexa poler ($a > 0{,}16$): $z = -0{,}4 \pm j\sqrt{a - 0{,}16} \Rightarrow$

$\Rightarrow a - 0{,}16 > 1 - 0{,}16 = 0{,}84 \Rightarrow a > 1$

välj alltså $\underline{\underline{-0{,}9 < a < -0{,}2}}$

3b. $$H(z) = \frac{z + 0{,}4}{z^2 + 0{,}8z - 0{,}2} =$$

$$= z \cdot \frac{1}{z^2+0{,}8z-0{,}2} + z^{-1} \cdot z \cdot \frac{0{,}4}{z^2+0{,}8z-0{,}2} =$$

$$= X_1(z) + X_2(z)$$

$\underline{X_1(z)}$: poler : $z = -0{,}4 \pm 0{,}6 = \begin{cases} 0{,}2 \\ -1 \end{cases} \Rightarrow$

$N' = 2z + 0{,}8 \Rightarrow$

$h_1[n] = A \cdot 0{,}2^n + B \cdot (-1)^n$

$$A = \left[\frac{T}{N'}\right]_{z=0{,}2} = \frac{1}{2 \cdot 0{,}2 + 0{,}8} = \frac{5}{6}$$

$$B = \left[\frac{T}{N'}\right]_{z=-1} = \frac{1}{2 \cdot (-1) + 0{,}8} = -\frac{5}{6}$$

$X_2(z)$ ger en fördröjning av $h_1[n]$:

$$\underline{\underline{h[n] = \frac{5}{6} \cdot 0{,}2^n - \frac{5}{6} \cdot (-1)^n + \frac{5}{6}\, 0{,}2^{n-1} \sigma[n-1] - \frac{5}{6} (-1)^{n-1} \sigma[n-1]}}$$

4.
$$y[n] = bx[n] + x[n-1] + y[n-1] + ay[n-2] \Rightarrow$$

$$Y(z)\cdot\left(1 - z^{-1} - az^{-2}\right) = \left(b + z^{-1}\right)X(z) \Rightarrow$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b + z^{-1}}{1 - z^{-1} - az^{-2}} = \underline{\underline{\frac{bz^2 + z}{z^2 - z - a}}}$$

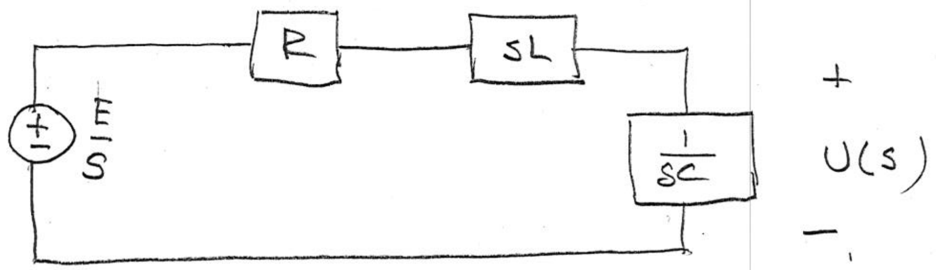
---

5. Högsta ingående frekvens är

$$f_{max} = \frac{6600\pi}{2\pi} Hz = 3300\,Hz \Rightarrow \underline{\underline{f_S > 6{,}6\,kHz}}$$

---

6. Komplext beräkningsschema $t > 0$



$R = 50\,\Omega$
$L = 0{,}25\,H$
$C = 625\,\mu F$
$E = 24\,V$

a

$$U(s) = \frac{\frac{1}{sC}}{R + sL + \frac{1}{sC}} \cdot \frac{E}{s} = \frac{1}{s^2LC + sRC + 1} \cdot \frac{E}{s} =$$

$$= \frac{E}{LC} \cdot \frac{1}{s\left(s^2 + s\frac{R}{L} + \frac{1}{LC}\right)} = 153600 \cdot \frac{1}{s\left(s^2 + 200s + 6400\right)} =$$

$$= 153600 \cdot \frac{T}{N}\,; \quad N' = 1\cdot(s^2 + 200s + 6400) + s\cdot(2s + 200)$$

poler $s = 0$; $s = -100 \pm \sqrt{10^4 - 6400} = -100 \pm 60 = \begin{cases} -160 \\ -40 \end{cases} \Rightarrow$

$$u(t) = A + B\cdot e^{-160t} + C\cdot e^{-40t}$$

$$A = 153600 \cdot \left[\frac{T}{N'}\right]_{s=0} = 24 \qquad B = 153600 \cdot \left[\frac{T}{N'}\right]_{s=-160} = 8$$

$$C = 153600 \cdot \left[\frac{T}{N'}\right]_{s=-40} = -32 \Rightarrow$$

$$\underline{\underline{u(t) = 24 + 8\cdot e^{-160t} - 32\cdot e^{-40t}\ V}}$$

---

b. $U(s) = \frac{E}{LC} \cdot \frac{1}{s \cdot \left(s^2 + s \cdot \frac{R}{L} + \frac{1}{LC}\right)}$

$u(t)$ är svängande om det finns komplexa poler:

$$s^2 + s \cdot \frac{R}{L} + \frac{1}{LC} = 0 \Rightarrow s = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

Komplexa poler om

$$\frac{R^2}{4L^2} - \frac{1}{LC} < 0 \Leftrightarrow \frac{R^2}{4L^2} < \frac{1}{LC} \Leftrightarrow$$

$$R^2 < \frac{4L}{C} = \frac{4 \cdot 0,25}{625 \cdot 10^{-6}} \Omega^2 = 1600 \Omega^2 \Rightarrow$$

$$\underline{\underline{R < 40 \Omega}}$$

7a. $T = 2s \Rightarrow \omega_0 = \frac{2\pi}{2} = \underline{\underline{\pi}}$ (rad/s)

$$c_0 = \frac{1}{T}\int_{1\,\text{period}} x(t)\,dt = \frac{1}{2}\int_{0,5}^{1,5}(-4)\,dt = \underline{\underline{-2}}$$

$k \neq 0$. Betrakta $y(t) = x(t) + 4$ som har samma c_k för $k \neq 0$.

$$c_k = \frac{1}{T}\int_{1\,\text{period}} y(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_{-0,5}^{0,5} 4\cdot e^{-jk\omega_0 t}\,dt =$$

$$= \frac{1}{T}\cdot 4\cdot\left[\frac{1}{-jk\omega_0}\cdot e^{-jk\omega_0 t}\right]_{-0,5}^{0,5} = \frac{4}{-jk\omega_0 T}\left(e^{jk0,5\omega_0} - e^{-jk0,5\omega_0}\right) =$$

$$= \frac{4}{-jk2\pi}\cdot\left(e^{jk\frac{\pi}{2}} - e^{-jk\frac{\pi}{2}}\right) = \frac{4}{-jk2\pi}\cdot\left(-2j\sin k\frac{\pi}{2}\right) =$$

$$= \frac{4}{k\pi}\cdot\sin k\frac{\pi}{2}$$

$$\left.\begin{aligned} a_k &= 2\cdot\mathrm{Re}\,c_k = \frac{8}{k\pi}\sin k\frac{\pi}{2} \\ b_k &= -2\cdot\mathrm{Im}\,c_k = 0 \end{aligned}\right\} \Longrightarrow$$

$a_0 = c_0 = \underline{\underline{-2}}$ $\qquad a_1 = \frac{8}{\pi}\cdot\sin\frac{\pi}{2} = \frac{8}{\pi} = \underline{\underline{2,55}}$

$a_2 = \frac{8}{\pi}\sin\pi = 0$

$b_1 = b_2 = \underline{\underline{0}}$

(7b) Stor fyrkant $T_1 = 2ms \Rightarrow f_{01} = 500 Hz$

Liten fyrkant $T_2 = 6ms \Rightarrow f_{02} = 167 Hz$

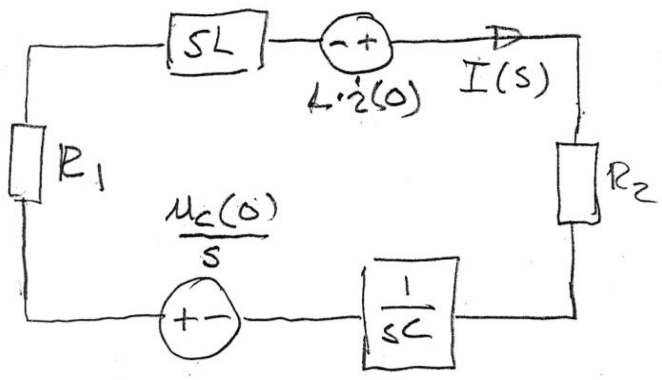
$\Rightarrow$ Ingående frekvenser är $k \cdot 500 Hz$; $k \cdot 167 Hz$; k=heltal

---

(8) Begynnelsevärden:

$$i(0) = \frac{E}{R_2} = \frac{16V}{80\Omega} = 0{,}2A \quad ; \quad u_c(0) = E = 16V$$

Beräkningsschema $t>0$:



$R_1 = 20\Omega$
$R_2 = 80\Omega$
$L = 10mH$
$C = 2\mu F$
$E = 16V$

$\Rightarrow$

$$I(s) = \frac{L \cdot i(0) + \frac{u(0)}{s}}{R_1 + R_2 + sL + \frac{1}{sC}} = \frac{s \cdot LC \cdot i(0) + C \cdot u(0)}{s^2 LC + s(R_1+R_2)C + 1} =$$

$$= \frac{s \cdot i(0) + \frac{1}{L} \cdot u(0)}{s^2 + s\frac{R_1+R_2}{L} + \frac{1}{LC}} = \frac{0{,}2s + 1600}{s^2 + 10^4 s + 50 \cdot 10^6} = \frac{T}{N}$$

$N' = 2s + 10^4$; poler: $s = -5000 \pm j5000 \Rightarrow$

$$i(t) = A \cdot e^{-5000t} \cdot \cos(5000t + \alpha)$$

$$A = 2 \cdot \left[\frac{T}{N'}\right]_{s=-5000+j5000} = 2 \cdot \frac{0{,}2(-5000+j5000)+1600}{-10000+j10000+10^4} = 0{,}233 \cdot e^{-j31^\circ}$$

$$\Rightarrow \quad \underline{\underline{i(t) = 0{,}233 \cdot e^{-5000t} \cdot \cos(5000t - 31^\circ)\ A}}$$

(9.) $H(s) = -\frac{Z_2}{R_1}$, där $Z_2 = R_2 // \left(R_3 + \frac{1}{sC}\right) =$

$$= \frac{R_2 \cdot \left(R_3 + \frac{1}{sC}\right)}{R_2 + R_3 + \frac{1}{sC}} = \frac{R_2(1 + sR_3C)}{1 + s(R_2+R_3)C} \Rightarrow$$

$$H(s) = -\frac{R_2}{R_1} \cdot \frac{1 + sR_3C}{1 + s(R_2+R_3)C} = -100 \cdot \frac{1 + s \cdot 3{,}3 \cdot 10^3 \cdot 10 \cdot 10^{-9}}{1 + s \cdot 223{,}3 \cdot 10^3 \cdot 10 \cdot 10^{-9}} =$$

$$= -100 \cdot \frac{1 + \frac{s}{30303}}{1 + \frac{s}{447{,}8}} \Rightarrow$$

$\underline{\underline{A = -100}}$ $\quad \underline{\underline{\omega_1 = 30303 \text{ rad/s}}}$ $\quad \underline{\underline{\omega_2 = 447{,}8 \text{ rad/s}}}$

10.

Bestäm $H(z)$:

$$Y(z) = X(z) + \sqrt{2}\, z^{-1} X(z) + z^{-2} X(z) - 0{,}95\sqrt{2}\, z^{-1} Y(z) - 0{,}9025 \cdot z^{-2} Y(z)$$

$$\Longrightarrow \quad H(z) = \frac{1 + \sqrt{2}\, z^{-1} + z^{-2}}{1 + 0{,}95\sqrt{2}\, z^{-1} + 0{,}9025\, z^{-2}} = \frac{z^2 + \sqrt{2}\, z + 1}{z^2 + 0{,}95\sqrt{2}\, z + 0{,}9025}$$

Nollställen:

$$z^2 + \sqrt{2} \cdot z + 1 = 0 \Longrightarrow$$

$$z = -\frac{\sqrt{2}}{2} \pm \sqrt{\frac{2}{4} - 1} = -\frac{1}{\sqrt{2}} \pm j\frac{1}{\sqrt{2}} = e^{\pm j135^\circ}$$

Filtret släcker frekvensen som motsvarar $\Omega = 135^\circ$.

$$\Omega = 2\pi \cdot \frac{F}{F_s} \Longrightarrow F = \frac{\Omega}{2\pi} \cdot F_s = \frac{135^\circ}{360^\circ} \cdot 1000\,\text{Hz} = \underline{\underline{375\,\text{Hz}}}$$

(11.) $Y(s) = H_1 \cdot (X(s) - H_2 \cdot Y(s)) \Longrightarrow$

$$H(s) = \frac{H_1(s)}{1+H_1(s)\cdot H_2(s)} = \frac{\frac{3}{s+1}}{1+\frac{3}{s+1}\cdot\frac{b}{s+2}} =$$

$$= \frac{3}{s+1+\frac{3b}{s+2}} = \frac{3(s+1)}{s^2+3s+2+3b}$$

Poler: $s = -1{,}5 \pm \sqrt{2{,}25-2-3b} = -1{,}5 \pm \sqrt{0{,}25-3b}$

Stabilt om: $0{,}25 - 3b < 1{,}5$, dvs

om $3b > -1{,}25 = -\frac{5}{4} \Leftrightarrow \underline{b > -\frac{5}{12}}$

Icke-svängande stegsvar om

$0{,}25 - 3b > 0 \Leftrightarrow 3b < \frac{1}{4} \Leftrightarrow \underline{b < \frac{1}{12}}$

Alltså $\underline{\underline{-\frac{5}{12} < b < \frac{1}{12}}}$

(12.) Bestäm frekvensfunktionen $H(\Omega)$:

$$H(\Omega) = \left[H(z)\right]_{z=e^{j\Omega}}$$

$$Y(z) = K \cdot X(z) - a z^{-1} Y(z) \Rightarrow H(z) = K \cdot \frac{z}{z+a} \Rightarrow$$

$$H(\Omega) = K \cdot \frac{e^{j\Omega}}{e^{j\Omega} + a} = K \cdot \frac{e^{j\Omega}}{\cos\Omega + a + j\sin\Omega}$$

Betrakta $|H(\Omega)|^2 = K^2 \cdot \frac{1}{(a+\cos\Omega)^2 + \sin^2\Omega} =$

$$= K^2 \cdot \frac{1}{a^2+1+2a\cos\Omega}$$

$|H(\Omega)|^2_{max}$ är strängt växande eller avtagande beroende på a, men LP-filter $\Rightarrow$ avtagande $\Rightarrow$

$$|H(\Omega)|^2_{max} = |H(0)|^2 = 1$$

övre gränsfrekvens $f_g = 200\,\text{Hz} \Rightarrow \Omega_g = 2\pi \cdot \frac{F_g}{F_S} =$

$= 2\pi \cdot \frac{200\,\text{Hz}}{1200\,\text{Hz}} = \frac{\pi}{3}$ rad. Vid $\Omega = \frac{\pi}{3}$ gäller att

$|H(\Omega)| = \frac{1}{\sqrt{2}}$ dvs $\left|H\left(\frac{\pi}{3}\right)\right|^2 = \frac{|H(\Omega)|^2_{max}}{2} \Rightarrow$

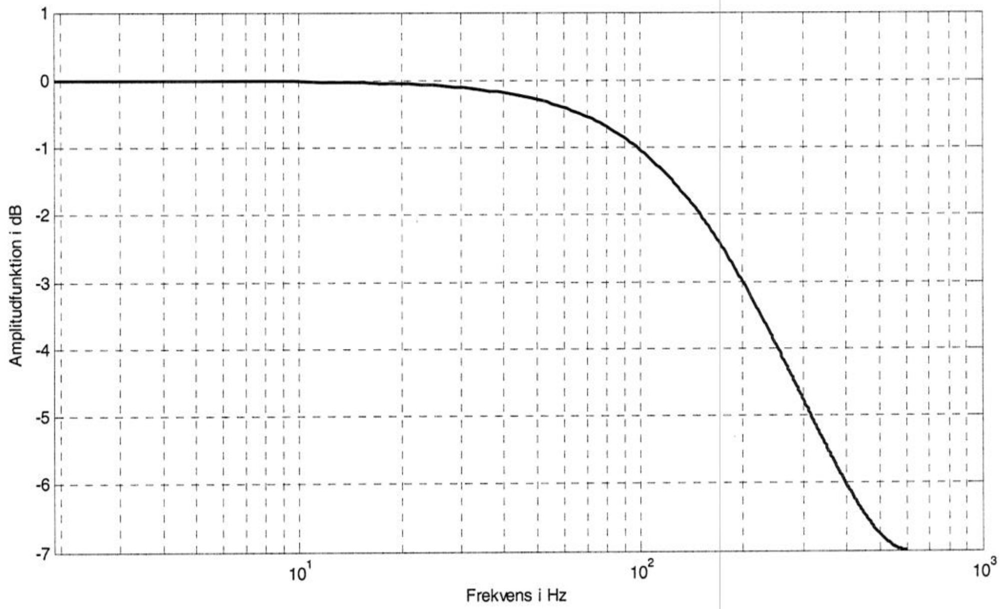
$$\frac{K^2}{a^2+1+2a\cos\frac{\pi}{3}} = \frac{K^2}{2(a^2+2a+1)} \Leftrightarrow$$

$$\frac{K^2}{a^2+1+a} = \frac{K^2}{2(a^2+2a+1)} \Rightarrow 2a^2+4a+2 = a^2+1+a \Rightarrow$$

$a^2+3a+1 = 0 \Rightarrow a = \frac{-3\pm\sqrt{5}}{2}$. Men minustecknet ger instabilt filter så $a = \frac{-3+\sqrt{5}}{2} = \underline{\underline{-0{,}38197}}$

$H(1) = 1 \Rightarrow 1 = K \cdot \frac{1}{1-0{,}38197} \Rightarrow \underline{\underline{K = 0{,}61803}}$

På nästa sida kan du se filtrets amplitudfunktion!



1
0
-1
-2
-3
-4
-5
-6
-7
Amplitudfunktion i dB
10^1
10^2
10^3
Frekvens i Hz