Maxwell's Equations

11

From a long view of the history of mankind—seen from say, ten thousand years from now—there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics. The American civil war will pale into provincial insignificance in comparison with this important scientific event of the same decade.

Richard P. Feynman (1918–1988)
Physicist, Nobel laureate

11.1 Introduction: The Electromagnetic Field

Most students in engineering and science have heard the term "Maxwell's equations," and some may also have heard that Maxwell's equations "describe all electromagnetic phenomena." However, it is not always clear what exactly do we mean by these equations. How are they any different from what we have studied in the previous chapters? We recall discussing static and time-dependent fields and, in the process, discussed many applications. To do so, we used the definitions of the curl and divergence of the electric and magnetic fields: what we called "the postulates." Are Maxwell's equations different? Do they add anything to the previously described phenomena? Perhaps the best question to ask is the following: Is there any other electromagnetic phenomenon that was not discussed in the previous chapters because the definitions we used were not sufficient to do so? If so, do Maxwell's equations define these yet unknown properties of the electromagnetic field? The answer to the latter is emphatically yes.

In fact, we do not need to go far to find applications which could not be treated using, for example, Faraday's law. The most obvious is transmission of power as, for example, in radio or television. All applications related to transmission of power (radar, communication, radio, etc.) were conspicuously missing in the previous chapters, but there is an even more important (and related) aspect of the electromagnetic field which was not mentioned until now. Take, for example, induction of voltage in a loop. Faraday's law gives an accurate statement of how the induction occurs and the magnitude of the induced emf. Now let's say that two loops are located a short distance from each other and one loop induces an electromotive force in the second. If we were to separate the loops a very long distance from each other and measure the induced voltage in the second loop, the magnitude will be very small. The question is, however, this: Is there any lag in time between switching on the current in the first loop and detection of the induced voltage in the second loop because of the distance between the loops? Faraday's law says nothing about that and neither do any of the postulates used previously. Intuitively, we know there must be a time lag since nothing can occur instantaneously. In this regard, consider the following: On January 22, 2003, NASA received the last transmission from the *Pioneer* 10 space probe. At that time, *Pioneer* 10 was 5 weeks shy of its 31st year of space flight and was over 12.2 billion km from the Earth.[1] At that distance, the transmission took approximately 11 h 18 min to reach the Earth. This is hardly instantaneous. In fact, if we divide distance by time, we find that the information has

[1] *Pioneer* 10 was launched on March 2, 1972, with the intention of moving out of the solar system. Designed for a mission of 21 months, it was the first spacecraft to pass through the asteroid belt and out of the solar system. The spacecraft carries a unique plaque that identifies the Earth and its inhabitants as the designers of the craft. *Pioneer* 10 is still flying, heading for the red star Aldebaran in the constellation Taurus where it is expected to arrive in about two million years. Stay tuned!

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traveled through space at the speed of light. It should then be obvious that something must be missing since we cannot account for this time lag when using Faraday's or Ampere's laws.

Maxwell's equations are, in fact, the four postulates we introduced in the previous chapter with a modification to Ampere's law to account for finite speed of propagation of power. This modification is rather simple and we usually refer to it as displacement current or displacement current density and is simply a statement of conservation of charge. However, in spite of its simplicity, it is far reaching and, as we shall see shortly, crucial to all applications involving transmission of signals or power. In a way, the remainder of this book is dedicated to the discussion of this aspect of electromagnetic fields.

To return now to the applications discussed in the previous chapters, we should ask ourselves the following question: If the postulates used need to be modified, do we also need to go back and discuss all that we have done and perhaps modify all previous relations? The answer, fortunately, is no. In all applications until now, there was no need for these modifications even though the solutions obtained were often only approximations. However, these were very good approximations and there would be very little to be gained by including the ideas introduced in this chapter into the previous results, as we shall see. For example, two coils, near each other, experience a force. The force is not instantaneous, but because the distance between the coils is small, the time lag is so small as to be justifiably neglected.

11.2 Maxwell's Equations

When, in 1873, James Clerk Maxwell[2] wrote his now famous *Treatise on Electricity and Magnetism*, he wrote in the preface to the book that his purpose was essentially that of explaining Faraday's ideas (published in *Experimental Researches in Electricity* in 1839) into a mathematical and, therefore, more universal form. He makes it amply clear that his treatise is a sort of summary or unification of the knowledge in electrical and magnetic fields as put forward by others, including those who preceded Faraday (Ampere, Gauss, Coulomb, and others). We might add that the notation we use today to write Maxwell's equations was introduced by Oliver Heaviside[3] almost 20 years after Maxwell's theory appeared. If you were to read Maxwell's book, you might not recognize the equations written in the previous chapters or in this. What then is Maxwell's unique contribution? Why do we normally refer to the electromagnetic field equations as "Maxwell's equations"? Surely, it is more than simply because he summarized what others have done.

His main contribution is in proposing the inclusion of displacement currents[4] in Ampere's law. This seemingly minor change in the field equations as known before his time was, in fact, a fundamental change in the theory of electromagnetics. Maxwell's ideas, which were often expressed in mechanical terms, were not immediately accepted since they implied a number of aspects of the electric and magnetic fields that had no proof at the time. Maxwell himself had no experimental

[2] James Clerk Maxwell (1831–1879), Scottish scientist, trained as a mathematician. Between 1856 and 1860 he lectured in Aberdeen and in 1860 became Professor of Natural Philosophy and Astronomy at King's College, London, until 1865. After that, he resigned and busied himself in writing, including on the *Treatise on Electricity and Magnetism*, published in 1873. Maxwell's work was not limited to electricity and magnetism. He wrote on the theory of gases, on heat, and on such topics as light, color, color blindness, the rings of Saturn, and others. The three-color combination (red, green, and blue) used to this day in defining color processes such as television and monitor screens was invented by Maxwell in 1855. Although a modest man, he knew the value of his work and was proud of it. The publication of the *Treatise on Electricity and Magnetism* was a turning point in electromagnetics. It was for the first time since Oersted's discovery of the link between electricity and magnetism that this link extended to the generation and propagation of waves. This was shown experimentally by Hertz 15 years later and opened the way to the invention of radio and the communication era.

[3] Oliver Heaviside (1850–1925). Oliver Heaviside was by all accounts the "enfant terrible" of electromagnetics. A brilliant man with a natural gift for mathematical analysis, he was the incarnation of antiestablishment. Heaviside dropped out of school at age 16 and at age 18 started working for the Anglo-Danish cable company. He worked for about 6 years during which he taught himself the theories of electricity and magnetism and, apparently, applied mathematics. After that, Heaviside set out to understand and explain Maxwell's theory, and in the process, he derived the modern form of Maxwell's equations (at about the same time Hertz did). He was one of the first to use phasors and did much to propagate the use of vector analysis. Heaviside may well be considered the developer of operational calculus and of the theory of transmission lines. Much of his work remains uncredited, a process which started in his lifetime. He was reclusive and abrasive, qualities that did not win him friends. Constant attacks on the "mathematicians of Cambridge" caused much friction, as did the fact that his papers were difficult to understand. For example, he insisted that potentials have no value. In his words, these were "Maxwell's monsters" and should be "murdered." Similarly, he dismissed the theory of relativity. With all his failings, quite a bit of what we study today in electromagnetics and circuits must be credited to Heaviside.

[4] The term displacement and displacement current were coined by Maxwell from the analogies he used. To Maxwell, flux lines were analogous to lines of flow in an incompressible fluid. Using this analogy, he called the quantity **D**, the electric displacement and therefore the current density produced by its time derivative, the displacement current density.

proof for the existence of displacement currents, but it is obvious from reading his book that he considered both displacement currents and the implications of their existence as fact. Experimental proof of the existence of electromagnetic waves at frequencies well below those of light came only in 1888, when the young Heinrich Hertz[5] showed through his famous experiments that an electromagnetic disturbance travels through air and can be received at a distance. This was almost 10 years after Maxwell died and 15 years after he wrote the Treatise. Some of the most important implications of displacement currents and of Maxwell's equations in general are as follows:

(1) Interdependence of the electric and magnetic fields.
(2) The existence of electromagnetic waves.
(3) Finite speed of propagation of electromagnetic waves.
(4) Propagation in free space is at the speed of light, and light itself is an electromagnetic wave.

11.2.1 Maxwell's Equations in Differential Form

To understand the importance of these concepts and Maxwell's contribution to the theory of electromagnetics, it is well worth looking at the electromagnetic field equations as they existed before Maxwell's introduction of displacement currents (below on the left) and after Maxwell's modification (below on the right):

Field equations before Maxwell's modification		Field equations after Maxwell's modification	
$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	(11.1)	$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	(11.5)
$\nabla \times \mathbf{H} = \mathbf{J} \quad [\text{A/m}^2]$	(11.2)	$\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t} \quad [\text{A/m}^2]$	(11.6)
$\nabla \cdot \mathbf{D} = \rho_v \quad [\text{C/m}^3]$	(11.3)	$\nabla \cdot \mathbf{D} = \rho_v \quad [\text{C/m}^3]$	(11.7)
$\nabla \cdot \mathbf{B} = 0$	(11.4)	$\nabla \cdot \mathbf{B} = 0$	(11.8)

Comparison between the two sets reveals that the only difference is in the last term of **Eq. (11.6)**. This term is a current density and is called the ***displacement current density***.

Perhaps the easiest way to show that the pre-Maxwell equations are not, in general, adequate is to show that they are not consistent with the continuity equation or that conservation of charge is not satisfied unless the displacement current density in **Eq. (11.6)** is introduced. To do so, we take the divergence on both sides of **Eq. (11.2)**:

$$\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \mathbf{J} \tag{11.9}$$

On the left-hand side, the divergence of the curl of a vector is identically zero. Thus, we get

$$\nabla \cdot \mathbf{J} = 0 \tag{11.10}$$

On the other hand, if we do the same with **Eq. (11.6)**, we get

$$\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \mathbf{J} + \nabla \cdot \left(\frac{\partial \mathbf{D}}{\partial t}\right) \tag{11.11}$$

[5] Heinrich Rudolph Hertz (1857–1894). Hertz was trained as an engineer but had considerable interest in other areas, including mathematics and languages. At the suggestion of Herman von Helmholtz, he undertook a series of experiments which, in 1888, led to verification of Maxwell's theory. This included proof of propagation of waves which he showed by receiving the disturbance produced by a spark with a receiver which was essentially a loop with a gap. In the process, he measured the speed of propagation and found it to be that of light. The conclusion that light itself was an electromagnetic wave was a logical extension from these experiments. At the age of 31, Hertz succeeded where others failed. In his experiments, he showed many of the properties of waves, including reflection, polarization, refraction, periodicity, resonance, and even the use of parabolic antennas. It is all the more tragic that he died only 6 years later at the age of 37. The notation we use in this book is due to Hertz (and Heaviside) rather than due to Maxwell. Unlike Maxwell, who emphasized the use of potentials, Hertz and Heaviside emphasized the use of the electric and magnetic fields.

Again, the left-hand side is identically zero and we can write

$$\nabla \cdot \mathbf{J} = -\frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) \tag{11.12}$$

where the time derivative and the divergence were interchanged. Substituting $\nabla \cdot \mathbf{D}$ from **Eq. (11.7)** gives

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \tag{11.13}$$

which is exactly the continuity equation [see **Eq. (7.32)**]. Thus, introduction of the displacement current density in **Eq. (11.2)** is equivalent to enforcing the law of conservation of charge as was discussed in **Section 7.6**. That this should be so is intuitively understood: If the law of conservation of charge is correct and if the field equations must obey this law, then the law must be incorporated into the equations. However, there is one question that we have alluded to in the introduction. If, indeed, the displacement current density must be included, how is it possible that all the results obtained in previous chapters were considered to be correct while explicitly neglecting the displacement current term? An even more important question is: What are the implications of the new results that the inclusion of displacement current suggests? The answer to the first question is, in fact, implicit in the continuity equation itself. If we assume that the time derivative of current density is zero, that is, that the charge density is constant with time, the continuity equation states that $\nabla \cdot \mathbf{J} = 0$. This corresponds to the pre-Maxwell equations. In other words, as long as we deal with steady (DC) currents or with zero currents (static charges), all relations we have developed in previous chapters are correct (and, in fact, exact). The answer to the second question is much more involved and will be answered gradually in the following chapters. At this point we simply note that as long as the displacement current is small [low value of the time derivative in **Eq. (11.6)** or **(11.13)** or, alternatively, low frequencies], the displacement current density may be neglected. A more quantitative explanation will follow in **Chapter 12**. We now give a very simple example that indicates the importance of displacement currents for understanding even the simplest aspects of electromagnetics.

Example 11.1 Displacement Current in a Capacitor Consider the capacitor in **Figure 11.1a**. The capacitor is connected to an AC source to form a closed circuit. Calculate the displacement current in the capacitor.

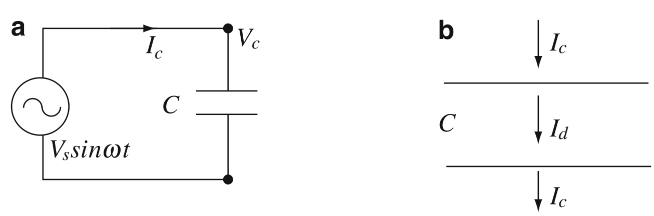

Figure 11.1 Displacement current in a capacitor. (**a**) Capacitor connected to an AC generator. (**b**) Relation between the conduction and displacement current

Solution: A current flowing in the circuit can be physically measured using an AC amperemeter. The current may be calculated using circuit concepts, and since current in a closed circuit is the same everywhere in the circuit, this must also be equal to the displacement current in the capacitor. A second method is to calculate the electric flux density **D** in the capacitor and then calculate $\partial \mathbf{D}/\partial t$ in the capacitor to obtain the displacement current density.

Method (1) The current in the circuit in **Figure 11.1a** is

$$I = C\frac{dV_c}{dt} = CV_s\omega\cos\omega t \quad [\text{A}]$$

Because this is the current at any point in the circuit, and a closed circuit must have the same current everywhere, this current must also exist inside the capacitor. There is no other way. We are therefore forced to allow for the existence of a current through the dielectric in the capacitor even though a dielectric cannot support a conduction current (after all, it is an insulator). Therefore, the displacement current in the dielectric must be equal to the conduction current in the circuit (**Figure 11.1b**); that is,

$$I_d = CV_s\omega\cos\omega t \qquad [\mathrm{A}]$$

Method (2) An alternative method is to calculate the electric flux density **D** in the capacitor and the displacement current from its time derivative. For a parallel plate capacitor, the electric field intensity between the plates is uniform and its magnitude is

$$E = \frac{V_c}{d} = \frac{V_s\sin\omega t}{d} \qquad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]$$

where V_c is the potential across the capacitor's plates and d the distance between the plates. Using this relation and $\mathbf{D} = \varepsilon\mathbf{E}$, the displacement current density is calculated using **Eq. (11.6)**:

$$J_d = \frac{\partial D}{\partial t} = \varepsilon\frac{\partial E}{\partial t} = \varepsilon\frac{\partial}{\partial t}\left(\frac{V_s\sin\omega t}{d}\right) = \frac{\varepsilon}{d}V_s\omega\cos\omega t \qquad \left[\frac{\mathrm{A}}{\mathrm{m}^2}\right]$$

To find the total displacement current, the current density must be integrated over the surface area of the plates of the capacitor. Denoting this area as S, we get

$$I_d = \int_{s'}\mathbf{J}_d\cdot d\mathbf{s}' = \frac{\varepsilon V_s\omega\cos\omega t}{d}\int_{s'}ds' = \frac{\varepsilon S V_s\omega\cos\omega t}{d} = CV_s\omega\cos\omega t \quad [\mathrm{A}]$$

where $C = \varepsilon S/d$ is the capacitance of a parallel plate capacitor of plate area S and distance d between the plates. This result is identical to that in Method (1).

Here, we have an example in which the displacement current is actually necessary to account for the behavior of the circuit. In its absence, we must assume that current cannot flow through the capacitor. In circuit theory, displacement currents are not normally used or assumed. To account for their effects, the common explanation is that the plates of the capacitor are alternately charged with positive and negative charges.

Example 11.2 A slab of perfect dielectric material ($\varepsilon_r = 2$) is placed in a microwave oven. The oven produces an electric field (as well as a magnetic field). Assume that the electric field intensity is uniform in the slab and sinusoidal in form and that it is perpendicular to the surface of the slab. The microwave oven operates at a frequency of 2.45 GHz (1 GHz = 10^9 Hz) and produces an electric field intensity with amplitude 500 V/m inside the dielectric:

(a) Calculate the displacement current density in the dielectric.
(b) Is there a displacement current in air? If so, calculate it.

Solution: Calculate the electric flux density D in the dielectric and calculate its derivative with respect to time to get the displacement current density. The displacement current only requires that an electric field exists and is always in the direction of the electric field intensity. Therefore, any material in which there is an electric field will support a displacement current, including free space:

(a) The electric field intensity and electric flux density in the dielectric are

$$E = E_0\sin\omega t \quad \rightarrow \quad D = \varepsilon_r\varepsilon_0 E_0\sin\omega t \qquad [\mathrm{V/m}]$$

The displacement current density in the dielectric is

$$J_d = \frac{\partial D}{\partial t} = \varepsilon_r \varepsilon_0 E_0 \omega \cos\omega t \qquad \left[\frac{\text{A}}{\text{m}^2}\right]$$

For the values given, this current density is

$$J_d = 2 \times 8.854 \times 10^{-12} \times 500 \times 2 \times \pi \times 2.45 \times 10^9 \times \cos\left(2 \times \pi \times 2.45 \times 10^9\right)t = 136.3\cos\left(4.9 \times 10^9 \pi\right)t \quad \left[\frac{\text{A}}{\text{m}^2}\right]$$

with a peak current density of 136.3 A/m^2.

(b) In air, the electric flux density D is the same as in the dielectric ($D_{1n} = D_{2n}$, although the electric field intensity is twice as high in air). Since D is the same, so must the displacement current density be the same. The displacement current density in air is

$$J_d = 136.3\cos\left(4.9 \times 10^9 \pi\right)t \quad \left[\frac{\text{A}}{\text{m}^2}\right].$$

11.2.2 Maxwell's Equations in Integral Form

Equations (11.5) through **(11.8)** are the differential or point form of Maxwell's equations. As such, they describe the fields in space and, as we shall see shortly, lead to partial differential equations. This form, however, is not always convenient. For the calculation of fields and field-related quantities, an integral expression is often more convenient and, sometimes, more descriptive of the phenomenon involved. It is therefore useful to obtain the integral forms of Maxwell's equations. This is done by integrating the two curl equations over an arbitrary open surface and the two divergence equations over an arbitrary volume. We start with **Eq. (11.6)** since the integral form of the remaining equations was obtained in **Chapters 4**, **8**, and **10**. Integrating over an arbitrary open surface s, we get

$$\int_s (\nabla \times \mathbf{H}) \cdot d\mathbf{s} = \int_s \mathbf{J} \cdot d\mathbf{s} + \int_s \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s} \tag{11.14}$$

The expression on the left-hand side is converted to a contour integral using Stokes' theorem. Ampere's law now becomes

$$\int_s (\nabla \times \mathbf{H}) \cdot d\mathbf{s} = \oint_C \mathbf{H} \cdot d\mathbf{l} = I_c + \int_s \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s} = I_c + I_d \tag{11.15}$$

The term on the left-hand side is the circulation of the magnetic field intensity, whereas on the right-hand side, the first term is the conduction current (all currents except displacement currents, including induced currents) and the second is the displacement current. The integral form of Maxwell's equations is therefore

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} \quad \text{(Faraday's law)} \tag{11.16}$$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_c + \int_s \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s} \quad \text{(Ampere's law)} \tag{11.17}$$

$$\oint_s \mathbf{D} \cdot d\mathbf{s} = Q \quad \text{(Gauss's law)} \tag{11.18}$$

$$\oint_s \mathbf{B} \cdot d\mathbf{s} = 0 \quad \text{(no monopoles)} \tag{11.19}$$

The first of these is Faraday's law as derived in **Chapter 10**. The second is Ampere's law. We first introduced this law in **Chapter 8** for the magnetic fields of steady currents. The law as given here is sometimes called the modified Ampere's law to distinguish it from the pre-Maxwell form defined in **Chapter 8**. The third relation is Gauss's law, which was discussed in **Chapter 4**. The fourth indicates the divergence-free condition of the magnetic flux density which was discussed at length in **Chapter 8** as indicating the fact that the magnetic field is always generated by a pair of poles (i.e., no single magnetic poles exist).

In practical applications, we may be required to solve for any or all of the variables in Maxwell's equations. It is well worth pausing here to discuss these equations. In particular, we ask ourselves if, indeed, these equations are all that we need to solve an electromagnetic problem.

First, we note that the equations [either in differential form in **Eqs. (11.5)** through **(11.8)** or in integral form in **Eqs. (11.16)** through **(11.19)**], contain four vector variables **E**, **D**, **B**, and **H** and two sources: **J** (or I) and ρ_v (or Q). The first is a vector source, whereas the second a scalar source. Each vector variable has three components in space, and, therefore, we actually have 12 unknown values for the 12 components of the fields. Since the first two equations are vector equations, they are equivalent to six scalar equations. The last two equations **[Eqs. (11.18)** and **(11.19)]** are scalar equations. Thus, we have 8 scalar equations in 12 unknowns. Clearly, some additional relations must be added in order to solve the equations. Before we add any relations, we must also ascertain if the four Maxwell's equations are independent. If they are not, additional relations might be required.

Recall the way that Maxwell's equations were derived. They were based on the definition of the curl and divergence. At no point did we require that the equations be independent. In fact, the last two equations in each set can be derived from the first two with the aid of the continuity equation. To see that this is the case, consider **Eq. (11.6)**. If we take the divergence on both sides of the equation, we get

$$\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \mathbf{J} + \nabla \cdot \frac{\partial \mathbf{D}}{\partial t} \tag{11.20}$$

The left-hand side is zero (the divergence of the curl of any vector is identically zero). If we interchange the time derivative and the gradient in the last term on the right-hand side, we get

$$0 = \nabla \cdot \mathbf{J} + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) \tag{11.21}$$

Now, from the continuity equation [**Eq.** (**11.13**)]

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \quad \rightarrow \quad 0 = -\frac{\partial \rho_v}{\partial t} + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) \tag{11.22}$$

and, finally,

$$\nabla \cdot \mathbf{D} = \rho_v \tag{11.23}$$

This is exactly **Eq. (11.7)**. A similar calculation shows that **Eq. (11.18)** can be derived from **Eq. (11.5)**.

This dependency of the equations means that, in fact, we only have two independent vector equations (the two curl equations) in four vector unknowns. Thus, only 6 scalar equations are available for solution and the number of unknowns is 12. We therefore need two more independent vector equations to solve the system. These equations are the two constitutive relations $\mathbf{B} = \mu\mathbf{H}$ and $\mathbf{D} = \varepsilon\mathbf{E}$. That this must be so can also be seen from the fact that Maxwell's equations as written in **Eqs. (11.5)** through **(11.8)** or **(11.16)** through **(11.19)** do not refer to material properties at all. On the other hand, we know that fields are very much dependent on materials. This dependency is expressed by the constitutive relations.

Table 11.1 Summary of the electromagnetic field equations in differential and integral forms

Maxwell's equations	Differential form	Integral form
Faraday's law	$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ (11.24)	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\dfrac{d\Phi}{dt}$ [V] (11.28)
Ampere's law	$\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$ [A/m^2] (11.25)	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_s \left(\mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{s}$ [A] (11.29)
Gauss's law	$\nabla \cdot \mathbf{D} = \rho_v$ [C/m^3] (11.26)	$\oint_s \mathbf{D} \cdot d\mathbf{s} = Q$ [C] (11.30)
No monopoles	$\nabla \cdot \mathbf{B} = 0$ (11.27)	$\oint_s \mathbf{B} \cdot d\mathbf{s} = 0$ (11.31)
Constitutive relations	$\mathbf{B} = \mu \mathbf{H}$ [T] (11.32) $\mathbf{D} = \varepsilon \mathbf{E}$ [C/m^2] (11.33)	
The Lorentz force equation	$\mathbf{F} = q(\mathbf{E} + \mathrm{v} \times \mathbf{B})$ [N] (11.34)	

In addition, the Lorentz force equation, first introduced in **Chapter 8**, should be considered as part of a complete set of equations required for the solution of an electromagnetic field problem. The complete set of equations is summarized in **Table 11.1**. Thus, a total of seven equations are normally considered to constitute a complete set. An additional constitutive relation was defined in **Eq. (7.12)** as $\mathbf{J} = \sigma \mathbf{E}$. This is not included in **Table 11.1** because it is limited to conducting regions and because it will be generalized in **Chapter 12**. In time-dependent applications of electromagnetics, only the first two Maxwell's equations are independent and need to be used for the solution, together with the constitutive relations. The force equation is included in the complete set because it cannot be derived from Maxwell's equations. For this reason, the above complete set is sometimes called the Maxwell–Lorentz equations.

Finally, because the material constitutive relations are an integral part of the electromagnetic field equations, it is worth reiterating the fact that material properties may be, in general, linear or nonlinear, isotropic or anisotropic, and homogeneous or nonhomogeneous. These properties were first defined in **Section 4.5.4.1** and are repeated below as a reminder:

Linear: Linearity in material properties (μ, ε, σ) means these properties do not change as the fields vary.
Homogeneous: Material properties do not depend on position: The material properties do not vary from point to point in space.
Isotropic: Material properties are independent of direction in space.

As an example, we may speak of nonlinear electromagnetic field equations if the equations are used in nonlinear media. In the following chapters, we will use mostly linear, isotropic, homogeneous media (simple media).

Example 11.3 Show that if $\mathbf{J} = 0$ in **Eq. (11.29)** and $Q = 0$ in **Eq. (11.30)**, the two divergence equations in **Eqs. (11.30)** and **(11.31)** can be obtained from **Eqs. (11.28)** and **(11.29)** without the need to invoke the continuity equation.

Solution: By taking the divergence of **Eqs. (11.24)** and **(11.25)**, we obtain the divergence equations in differential form. Integration on both sides of the result gives the answer:

$$\nabla \cdot (\nabla \times \mathbf{E}) = -\nabla \cdot \frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial (\nabla \cdot \mathbf{B})}{\partial t}$$

where the time derivative and the divergence were interchanged since these operations are mutually exclusive. Because the divergence of the curl of any vector field is identically zero, it follows that

$$\nabla \cdot \mathbf{B} = 0$$

Taking the volume integral of this relation and using the divergence theorem, we get

$$\int_v (\nabla \cdot \mathbf{B}) dv = \oint_s \mathbf{B} \cdot d\mathbf{s} = 0 \quad \rightarrow \quad \oint_s \mathbf{B} \cdot d\mathbf{s} = 0$$

This is **Eq. (11.31)**.

Similarly, starting with **Eq. (11.25)**, setting $\mathbf{J} = 0$, and taking the divergence on both sides, we get

$$\nabla\cdot(\nabla\times\mathbf{H}) = 0 = \nabla\cdot\frac{\partial\mathbf{D}}{\partial t} = \frac{\partial(\nabla\cdot\mathbf{D})}{\partial t} \quad\rightarrow\quad \nabla\cdot\mathbf{D} = 0$$

Again taking the volume integral and using the divergence theorem gives

$$\int_v (\nabla\cdot\mathbf{D})dv = \oint_s \mathbf{D}\cdot d\mathbf{s} = 0 \quad\rightarrow\quad \oint_s \mathbf{D}\cdot d\mathbf{s} = 0$$

The latter is identical to **Eq. (11.30)** for $Q = 0$.

11.3 Time-Dependent Potential Functions

The concept of potential functions was introduced in **Section 8.7** where the magnetic scalar and vector potentials were discussed. Also, the electric scalar potential was discussed at length in **Chapter 4**. The utility of these potential functions in the calculation of the electric and magnetic fields was shown and this utility was the justification of deriving the potentials in the first place. Here, we revisit the idea of scalar and vector potentials, but now the potentials are time-dependent although their purpose is still the same: to allow alternative, often simpler calculation of field quantities. We will also show that the potentials, and in particular the magnetic vector potential, lead to second-order partial differential equations representation of Maxwell's equations. The reason for pursuing this representation is the hope that by doing so, we may be able to find solutions to the field equations based on standard differential equations methods.

We recall that for a vector function to be represented by a scalar function alone, the vector function must be curl-free; that is,

$$\text{If}\ \ \nabla\times\mathbf{F} = 0 \quad\rightarrow\quad \mathbf{F} = -\nabla\Omega \tag{11.35}$$

where Ω is a scalar function and is called a scalar potential in the context of electromagnetics. Similarly, for a vector field to be represented by an auxiliary vector function, the vector field must be divergence-free:

$$\text{If}\ \ \nabla\cdot\mathbf{F} = 0 \quad\rightarrow\quad \mathbf{F} = \nabla\times\mathbf{W} \tag{11.36}$$

The vector function **W** is now a vector potential.

Vector and scalar potentials may be used even if the vector field is neither curl-free nor divergence-free. To do so, we invoke the Helmholtz theorem and write the vector field as the sum of an irrotational term and a solenoidal term (see **Section 2.5.1**):

$$\mathbf{G} = -\nabla U + \nabla\times\mathbf{C} \tag{11.37}$$

where U is a scalar potential and **C** a vector potential. The first term (the gradient of U) is irrotational since taking its curl yields zero. The second term is solenoidal since taking its divergence yields zero. Thus, the general process of defining scalar and vector potentials for vector fields is as follows:

(1) If the vector field is curl-free (irrotational), a scalar potential may be defined which completely describes the vector field.
(2) If the vector field is divergence-free (solenoidal), a vector potential may be defined which completely describes the vector field.
(3) For a general vector field, both a scalar and a vector potential are required to describe the vector field. The gradient of the scalar potential is used to describe the irrotational part of the field, whereas the vector potential is used to describe the solenoidal part of the field.

The potentials we define need not have any physical meaning, although they often do. Their definition is based on the vector properties of the fields and may be viewed as transformations. As such, as long as the transformation is unique and is properly defined, the potentials are valid. We will discuss here the electric scalar potential and the magnetic vector potential; these are needed for our discussion of electromagnetic fields. There are, however, many other potential functions that may be defined. We will only touch on some of these as examples.

11.3.1 Scalar Potentials

Regarding Maxwell's equations above, there are two scalar potentials that may be defined: the electric scalar potential and the magnetic scalar potential. However, inspection of Maxwell's equations shows that the equations are not, in general, curl-free. Therefore, scalar potentials cannot be used to solve for general electric and magnetic fields. There are, however, two situations in which scalar potentials may be used:

(1) If the time derivative of the magnetic flux density in Faraday's law **[Eq. (11.24)]** is zero, the electric field intensity is curl-free, and the electric potential may be used in lieu of the electric field intensity:

$$\mathbf{E} = -\nabla V \quad \text{if} \quad \nabla \times \mathbf{E} = 0 \tag{11.38}$$

(2) If the displacement current density and the conduction current density in Ampere's law **[Eq. (11.25)]** are zero, the magnetic field intensity is curl-free, and the magnetic scalar potential may be used in lieu of the magnetic field intensity:

$$\mathbf{H} = -\nabla \psi \quad \text{if} \quad \nabla \times \mathbf{H} = 0 \tag{11.39}$$

The electric scalar potential V and the magnetic scalar potential ψ can only be used by themselves for static electric and magnetic fields because only under static conditions can the electric and magnetic field intensity be curl-free. However, they are also useful in general electromagnetic fields in combination with vector potentials, as we shall see later.

11.3.2 The Magnetic Vector Potential

The magnetic vector potential was defined in **Section 8.7.1** as

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \text{because} \quad \nabla \cdot \mathbf{B} = 0 \tag{11.40}$$

This also applies to time-dependent magnetic fields because the definition is based entirely on the divergence-free condition of the magnetic flux density. The magnetic vector potential may be used in many ways, one of which was given in **Chapter 8**, where the Biot–Savart law in terms of the magnetic vector potential was used. Here, we will use the function to represent Maxwell's equations and, in the process, to show that it may be used for calculation of field quantities. To do so, we substitute the definition into Maxwell's first and second equations:

$$\nabla \times \mathbf{E} = -\frac{\partial(\nabla \times \mathbf{A})}{\partial t} \tag{11.41}$$

$$\nabla \times \frac{1}{\mu} \nabla \times \mathbf{A} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{11.42}$$

From **Eq. (11.41)** by interchanging the time derivative with the ∇ operator, we write

$$\nabla \times \mathbf{E} = \nabla \times \left(-\frac{\partial \mathbf{A}}{\partial t}\right) \quad \rightarrow \quad \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0 \tag{11.43}$$

Now the term in the parentheses is curl-free and it may be written as the gradient of the electric scalar potential:

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0 \quad \rightarrow \quad \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = -\nabla V \tag{11.44}$$

Rearranging this expression gives the general electric field intensity as

$$\boxed{\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla V \qquad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]} \tag{11.45}$$

This form is exactly that required by the Helmholtz theorem [**Eq. (11.37)**]. This, when substituted back into Faraday's law, gives the same relation as in **Eq. (11.43)** because $\nabla \times (\nabla V) = 0$. Also, the expression gives the correct result for the static electric field for which $\mathbf{E} = -\nabla V$. Now, we substitute the electric field intensity from **Eq. (11.45)** into **Eq. (11.42)**:

$$\nabla \times \frac{1}{\mu}\nabla \times \mathbf{A} = \mathbf{J} + \frac{\partial}{\partial t}\varepsilon\left(-\frac{\partial \mathbf{A}}{\partial t} - \nabla V\right) \tag{11.46}$$

For simplicity, we will assume that the material in which this relation is defined is linear, isotropic and homogeneous such that the permeability μ and permittivity ε are independent of position. This gives

$$\nabla \times (\nabla \times \mathbf{A}) = \mu\mathbf{J} - \mu\varepsilon\frac{\partial^2 \mathbf{A}}{\partial t^2} - \mu\varepsilon\frac{\partial}{\partial t}(\nabla V) \tag{11.47}$$

The left-hand side of **Eq. (11.47)** can be expanded using the identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$ [see **Eq. (2.136)**]. Substituting this and rearranging terms, we get

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} = \mu\mathbf{J} - \mu\varepsilon\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\mu\varepsilon\frac{\partial V}{\partial t}\right) \tag{11.48}$$

Inspection of **Eq. (11.48)** as well as the process leading to it reveals that Maxwell's equations were now replaced by a single equation in terms of the magnetic vector potential and the electric scalar potential. In fact, since the magnetic vector potential has three components, the equation is equivalent to three scalar equations. However, we need one more relation to take into account the electric scalar potential V. One possibility is to assume that V is zero. The second is to assume it is independent of time and the third is to assume it is constant in space. None of these is a general property of the electric field, and, therefore, we cannot assume these in general, although one of these may be occasionally used depending on the application. To resolve this difficulty, we use the fact that the divergence of **A** has not yet been defined. Since a vector is only uniquely defined if both its curl and its divergence are specified, we are free to choose the divergence of the magnetic vector potential. The second relation needed is therefore the divergence of the magnetic vector potential. From **Eq. (11.48)**, we note that if we choose

$$\boxed{\nabla \cdot \mathbf{A} = -\mu\varepsilon\frac{\partial V}{\partial t}} \tag{11.49}$$

the last term in **Eq. (11.48)** disappears and we get

$$\boxed{-\nabla^2\mathbf{A} = \mu\mathbf{J} - \mu\varepsilon\frac{\partial^2 \mathbf{A}}{\partial t^2}} \tag{11.50}$$

and this equation is now the equivalent form of Maxwell's equations; that is, instead of solving Maxwell's equations for the magnetic field intensity and the electric field intensity, we can solve for the magnetic vector potential and then obtain the magnetic and electric field intensities as well as flux densities from the magnetic vector potential.

The relation in **Eq. (11.49)** is called the ***Lorenz condition*** or the ***Lorenz gauge***. There are three questions associated with **Eqs. (11.49)** and **(11.50)**. First, is the Lorenz gauge the only possible choice? Second, how do we know that this choice is correct? Third, why should we use the magnetic vector potential in the first place since we can obtain a second-order equation in terms of **H** or **E**, as will be shown shortly? The answer to the first question is no. There are other choices that may be used, but this particular choice eliminates the scalar potential in the equation and therefore simplifies the equation which, in turn, also should simplify its solution. A commonly used gauge, particularly in static applications, is $\nabla \cdot \mathbf{J} = 0$, which is called the ***Coulomb's gauge*** and was introduced in **Chapter 8 [Eqs. (8.37)** and **(8.38)]**. The answer to the second question is that this choice is "consistent with the field equations." The latter statement means that Lorenz's condition is consistent with the principle of conservation of charge. The answer to the third question is twofold: First, it allows representation in terms of a single field variable **A**, instead of the need for **E** and **H**. Second, and perhaps more important, the magnetic vector potential is sometimes more convenient to use than the electric field intensity **E** or the magnetic field intensity **H**. While it is not the purpose here to prove this, it should be noticed that the magnetic vector potential is always in the direction of the current density **J**. This means that if the current density has a single component in space, the magnetic vector potential also has a single component. On the other hand, the magnetic field intensity has two components (perpendicular to the current). Without actually solving the equations, it is intuitively understood that solving for a single component of a field in space should be easier than solving for two components.

11.3.3 Other Potential Functions

By now, it should be understood that a potential function can be defined based on the properties of the original field, for the purpose of replacing the field with an equivalent but perhaps simpler representation. Many other potential functions may be defined in addition to the magnetic vector potential and the electric scalar potential discussed here. In **Section 11.3.1**, we defined the magnetic scalar potential in current-free regions. A current vector potential for steady currents in conducting media may be defined in a manner similar to the magnetic vector potential using the condition $\nabla \cdot \mathbf{J} = 0$ (see **Exercise 11.1**). Other potentials are the Hertz, Lorentz, and Whittaker potentials. However, because these potentials are not required for the development of the concepts presented in this book, we do not pursue these here (but see **Exercise 11.1** and **Problems 11.20, 11.21, 11.23, 11.24,** and **12.6**).

Example 11.4 Vector and Scalar Potentials in Conducting Media It is required to define the electromagnetic field equations for low frequencies in a highly conductive material in terms of the magnetic vector potential **A**. The material may be assumed to be linear, homogeneous, and isotropic in all its material properties. Show that by using Coulomb's gauge ($\nabla \cdot \mathbf{A} = 0$), the field equations reduce to:

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J}$$

Solution: The field equations are manipulated as in **Eqs. (11.41)** through **(11.48)**. This results in

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu \mathbf{J} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\mu\varepsilon \frac{\partial V}{\partial t}\right)$$

To simplify this equation, we must find a way of eliminating the first term on the left-hand side and the last term on the right-hand side. The first is obtained by substituting Coulomb's gauge:

$$-\nabla^2 \mathbf{A} = \mu \mathbf{J} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\mu\varepsilon \frac{\partial V}{\partial t}\right)$$

Now, we note that in a highly conducting material, the electric field intensity **E** is very low (the electric field intensity is zero in a perfectly conducting medium). From **Eq. (11.45)**, we conclude that this can happen only if both $\partial A/\partial t$ and ∇V are small. If we rewrite the field equation as

$$-\nabla^2 \mathbf{A} = \mu \mathbf{J} + \mu\varepsilon \frac{\partial}{\partial t}\left(-\frac{\partial \mathbf{A}}{\partial t} - \nabla V\right)$$

the term in parentheses is the electric field intensity [see **Eq. (11.45)**] and because both ∇V and $\partial A/\partial t$ are small, the entire second term on the right-hand side may be neglected, leading to the result:

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J}$$

Exercise 11.1 Vector and Scalar Potential Functions A vector potential **T** is defined in a conducting material with steady current ($\nabla \cdot \mathbf{J} = 0$) as

$$\nabla \times \mathbf{T} = \mathbf{J}, \quad \nabla \cdot \mathbf{T} = \sigma\mu \frac{\partial \psi}{\partial t}, \quad \text{and} \quad \mathbf{H} = \mathbf{T} - \nabla\psi$$

where ψ is a scalar function:

(a) If displacement currents are neglected and the only currents are induced currents, show from Maxwell's equations that

$$\nabla^2 \mathbf{T} = \sigma\mu \frac{\partial \mathbf{T}}{\partial t}$$

(b) What would you call the potentials **T** and ψ?

Answer **(b)** **T** is a current vector potential (also called an electric vector potential) and ψ is a magnetic scalar potential.

11.4 Interface Conditions for the Electromagnetic Field

We now return to a question we asked earlier: What happens to the electromagnetic field at the interface between two different materials? In the application of Maxwell's equations to various problems, we often encounter interfaces between different materials, with different material properties. The constitutive relations for the electric and magnetic fields indicate that fields are different in different materials. This, of course, is not new: The electrostatic field and the magnetostatic field were shown to behave differently in different materials. The interface conditions for the general, time-dependent electromagnetic fields as described by Maxwell's equations are essentially those we used for the static electric and magnetic fields together with any added condition that time dependency and the addition of the displacement current density in Ampere's law might add. In fact, we find that the modifications necessary are minor.

To define interface conditions, we must apply Maxwell's equations for the general electromagnetic field at the interface. From **Eqs. (11.1)** through **(11.8)**, **Table 11.1**, and the discussion in **Section 11.2**, it is clear that the only equations that have changed due to the time-dependent nature of the electromagnetic field are Faraday's law (discussed in **Chapter 10**) and Ampere's law (through the addition of displacement current). To see what needs to be done, consider **Table 11.2**, which lists the electrostatic and magnetostatic field equations together with the interface conditions we obtained in **Sections 4.6** and **9.3**, as well as Maxwell's equations. This table clearly indicates that the interface conditions on the normal components of the magnetic flux density and electric flux density did not change since the equations that defined them did not change. However, the fields D_{1n}, D_{2n}, B_{1n}, and B_{2n} as well as the charge density ρ_s are now time-dependent quantities. The interface conditions for the tangential components of the electric and magnetic field intensities need to be derived anew because the equations have been modified. This need is indicated by question marks in **Table 11.2**. This is our next task.

Table 11.2 Summary of the electromagnetic field equations and interface conditions

Static field equations	Interface conditions for static fields	Maxwell's equations in integral form	Interface conditions for time-dependent fields
$\oint_C \mathbf{E}\cdot d\mathbf{l} = 0$	$E_{1t} = E_{2t}$	$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d\Phi}{dt}$ [V]	?
$\oint_C \mathbf{H}\cdot d\mathbf{l} = I_{enclosed}$ [A]	$\hat{\mathbf{n}}\times(\mathbf{H}_1-\mathbf{H}_2)=\mathbf{J}_s$ [A/m] or $H_{1t}-H_{2t}=J_s^*$ [A/m]	$\oint_C \mathbf{H}\cdot d\mathbf{l} = \oint_s\left(\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}\right)\cdot d\mathbf{s}$ [A]	?
$\oint_s \mathbf{D}\cdot d\mathbf{s} = Q$ [C]	$\hat{\mathbf{n}}\cdot(\mathbf{D}_1-\mathbf{D}_2)=\rho_s$ [C/m^2] or $D_{1n}-D_{2n}=\rho_s$ [C/m^2]	$\oint_s \mathbf{D}\cdot d\mathbf{s} = Q$ [C]	$\hat{\mathbf{n}}\cdot(\mathbf{D}_1-\mathbf{D}_2)=\rho_s$ [C/m^2] or $D_{1n}-D_{2n}=\rho_s$ [C/m^2]
$\oint_s \mathbf{B}\cdot d\mathbf{s} = 0$	$B_{1n}=B_{2n}$	$\oint_s \mathbf{B}\cdot d\mathbf{s} = 0$	$B_{1n}=B_{2n}$

*This form requires the use of the right-hand rule to establish the vector relation between the tangential fields and the current density

Interface conditions are derived as follows:

(1) Two general materials are assumed to be in contact, forming an interface. By an interface, we mean an infinitely thin boundary, with no properties of its own. On each side of the interface, the materials have uniform properties and are linear. The interface may contain surface current densities and surface charge densities.
(2) Maxwell's equations in integral form are applied to the interface. From these, we find the conditions on the electric and magnetic fields on both sides of the interface since the fields must satisfy Maxwell's equations everywhere.
(3) Conditions at the interface are given in terms of components of the field at the interface. The tangential and normal components at the interface are the required interface conditions.
(4) The interface condition for the tangential components of **E** is derived from **Eq. (11.28)**.
(5) The interface condition for the tangential components of **H** is derived from **Eq. (11.29)**.

11.4.1 Interface Conditions for the Electric Field

To derive the interface conditions, we assume two different materials with properties as shown in **Figure 11.2a**. We assume a surface charge density ρ_s at the interface between the two different materials and apply **Eq. (11.28)**. Because **Eq. (11.28)** is a closed contour integral, we choose a closed contour that includes the interface alone, with two sides parallel to the interface and two sides perpendicular to the interface. The integration is in the direction shown. The closed contour is formed by the segments *ab*, *bc*, *cd*, and *da*:

$$\oint_{abcda} \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_{abcda}}{dt} = 0 \tag{11.51}$$

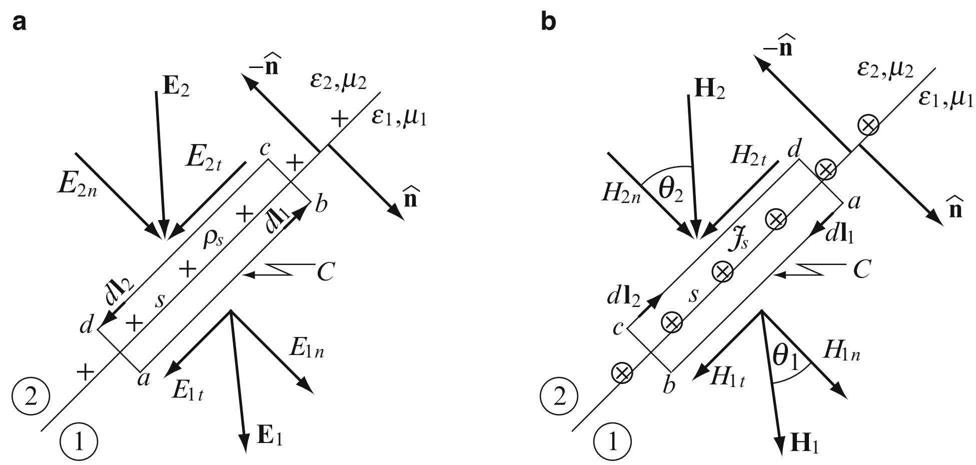

Figure 11.2 (**a**) Conditions for the tangential components of **E** at a general interface. (**b**) Conditions for the tangential components of **H** at a general interface

We note the following:

(1) The total flux Φ enclosed in the loop *abcda* is zero because, in the limit, the area enclosed by the loop is zero. This follows from the requirement that the contour only enclose the interface itself.

(2) The distances *bc* and *da* also tend to zero. Thus, the contribution of these two segments of the contour to the closed contour integration must be zero.

(3) The electric field in material (2) is in the direction of integration (in the direction of $d\mathbf{l}$); in material (1), the electric field intensity is in the direction opposite the direction of integration.

From these considerations we can now write **Eq. (11.51)** as

$$\oint_{abcda} \mathbf{E} \cdot d\mathbf{l} = \int_{ab} \mathbf{E}_1 \cdot d\mathbf{l}_1 + \int_{cd} \mathbf{E}_2 \cdot d\mathbf{l}_2 = 0 \tag{11.52}$$

Since the tangential component of E_2 is in the direction $d\mathbf{l}_2$ and the tangential component of E_1 is in the negative $d\mathbf{l}_1$ direction, we get

$$-\int_{ab} E_{1t} d\mathbf{l}_1 + \int_{cd} E_{2t} d\mathbf{l}_2 = 0 \tag{11.53}$$

By choosing the distance $ab = cd$, we get

$$\boxed{E_{1t} = E_{2t} \quad \text{or} \quad \frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}} \tag{11.54}$$

where $D_{1t} = \varepsilon_1 E_{1t}$ and $D_{2t} = \varepsilon_2 E_{2t}$.

The following should be noted from these relations:

(1) The tangential component of the electric field intensity is continuous across an interface between two general materials, regardless of charge densities on the surface.
(2) The tangential component of the electric flux density is discontinuous across the interface. The discontinuity is equal to the ratio between the permittivities of the materials.
(3) Interface conditions for the electric field intensity are independent of the magnetic field.
(4) The interface conditions for the time-dependent electric field are identical to those for the static electric field as given in **Chapter 4**. This is a consequence of Faraday's law: The connection between the electric and the magnetic fields is through the flux and the total flux on the interface is always zero (the interface has "zero" area).

11.4.2 Interface Conditions for the Magnetic Field

An almost identical sequence follows for the evaluation of the interface conditions for the magnetic field. The same interface between two general materials as in **Figure 11.2a** is used except that now, because we use Ampere's law, there are no charge densities on the interface but, rather, a current density. The conditions for application of Ampere's law are shown in **Figure 11.2b**. The following conditions are used:

(1) The contour *abcda* encloses the interface alone; that is, *bc* and *da* tend to zero and the area of the contour tends to zero.
(2) The tangential component of the magnetic field intensity in material (1) is in the same direction as the direction of integration, whereas in material (2), it is in the opposite direction.
(3) The total current enclosed by the contour is equal to the (surface) current density on the surface multiplied by the length *ab* (or *cd*, since these may be taken to be equal).

From these, we can write

$$\oint_c \mathbf{H}\cdot d\mathbf{l} = \int_{abcda} \mathbf{H}\cdot d\mathbf{l} = \int_{ab} \mathrm{H}_{1t}dl_1 - \int_{cd} H_{2t}dl_2 = \int_{ab} J_s dl \tag{11.55}$$

Note that in **Eq. (11.29)**, the integration for current is a surface integration because, in general, **J** is a current distributed over a volume. However, here, the current is distributed over a surface; therefore, the integration is on the line *ab*. The closed contour integral of $\mathbf{H}\cdot d\mathbf{l}$ always equals the current enclosed by the contour. Note also that the contribution to the line integral due to displacement current densities is zero. This can be best understood from the fact that as we approach the interface, the area enclosed by the contour *abcda* tends to zero, and, therefore, the surface integral over the volume current density $\partial\mathbf{D}/\partial t$ tends to zero. Choosing $ab = cd$, we get

$$\boxed{H_{1t} - H_{2t} = J_s \quad \text{and} \quad \frac{B_{1t}}{\mu_1} - \frac{B_{2t}}{\mu_2} = J_s \qquad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]} \tag{11.56}$$

where $\mathbf{B} = \mu\mathbf{H}$ was used to obtain the second relation from the first.

The interface conditions for the magnetic field may be summarized as follows:

(1) The tangential component of the magnetic field intensity is discontinuous in the presence of surface current densities. The discontinuity is equal to the surface current density. In the absence of surface current densities, the tangential component of the magnetic field intensity is continuous across the interface.
(2) The tangential component of the magnetic flux density is discontinuous.

The interface condition in **Eq. (11.56)** assumes the magnetic field intensity only has one tangential component or that each tangential component is treated separately. We have faced the same issue in **Section 9.3.1**. To establish a more general relation, we note from **Figure 11.2** that the tangential components of the magnetic field intensity may be written as $\mathbf{H}_{1t} = \hat{\mathbf{n}}\times\mathbf{H}_1$ [A/m] and $\mathbf{H}_{2t} = -\hat{\mathbf{n}}\times\mathbf{H}_2$ [A/m] where $\hat{\mathbf{n}}$ points into material (1) [see **Eq. (9.42)**]. With these observations, **Eq. (11.56)** is written as [see **Eq. (9.42)**]:

$$\boxed{\hat{\mathbf{n}}\times(\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \qquad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]} \quad \text{or} \quad \boxed{\hat{\mathbf{n}}\times\left(\frac{\mathbf{B}_1}{\mu_1} - \frac{\mathbf{B}_2}{\mu_2}\right) = \mathbf{J}_s \qquad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]} \tag{11.57}$$

As was discussed in **Section 9.3.1**, this form guarantees the correct magnitude and direction of the fields without resorting to the right-hand rule. **Equation (11.57)** should be used in all instances, although, when the magnetic field intensities have only one tangential component, **Eq. (11.56)** is equally suitable. These interface conditions are identical to the conditions we obtained for the magnetostatic field in **Chapter 9**. The conditions obtained for general materials are summarized in **Table 11.3**. There are a total of eight interface conditions, although only the four relations in the first and third row of **Table 11.3** were obtained from Maxwell's equations directly. The other four were obtained from the constitutive relations. But, do we need all these relations or, more importantly, are all these relations independent relations?

In **Section 11.2.2**, we mentioned that the two divergence equations can always be derived from the two curl equations. Gauss's law can be derived from Ampere's law and the equation of continuity and the zero divergence condition for the magnetic flux density can be derived from Faraday's law and the equation of continuity. Therefore, the last two interface conditions (those derived from the divergence equations) are not independent conditions. This means that to specify the continuity of the tangential electric field intensity and the continuity of the normal magnetic flux density is equivalent: The two can be derived from the same equations. Clearly, there is no need to specify both, and if we do, this may lead to overspecification. Similarly, the conditions for the tangential component of the magnetic field intensity and the normal component of the electric flux density are equivalent and only one should be specified.

Important Note: The electric and magnetic fields are mutually dependent on each other only in time-dependent cases. This also applies to interface conditions. The static electric and magnetic fields are independent of each other and we are therefore free to specify any and all boundary conditions.

Table 11.3 Electromagnetic interface conditions for general materials

	Electric field	Magnetic field
Tangential components	$E_{1t} = E_{2t}$	$\hat{\mathbf{n}} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$ [A/m] or $H_{1t} - H_{2t} = J_s$* [A/m]
	$\dfrac{D_{1t}}{\varepsilon_1} = \dfrac{D_{2t}}{\varepsilon_2}$	$\hat{\mathbf{n}} \times \left(\dfrac{\mathbf{B}_1}{\mu_1} - \dfrac{\mathbf{B}_2}{\mu_2}\right) = \mathbf{J}_s$ [A/m] or $\dfrac{B_{1t}}{\mu_1} - \dfrac{B_{2t}}{\mu_2} = J_s$ $\left[\dfrac{\text{A}}{\text{m}}\right]$
Normal components	$\hat{\mathbf{n}} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$ [C/m^2] or $D_{1n} - D_{2n} = \rho_s$ [C/m^2]	$B_{1n} = B_{2n}$
	$\hat{\mathbf{n}} \cdot (\varepsilon_1 \mathbf{E}_1 - \varepsilon_2 \mathbf{E}_2) = \rho_s$ [C/m^2] or $\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$ [C/m^2]	$\mu_1 H_{1n} = \mu_2 H_{2n}$

* This form requires the use of the right-hand rule to establish the vector relation between the tangential componenets and the current density

The interface conditions as discussed here are for two general materials. Because of this, we had to include both surface charge densities and surface current densities in the conditions. As we have seen in **Chapters 4** and **9**, a surface charge density may exist on the surface of a dielectric due to polarization or due to physical charges being placed or generated on the surface (for example, by friction). Another possible source of charges at an interface is due to flow of current across the interface between lossy dielectrics (see **Section 7.8**). Surface charges may also exist at the interface between perfect conductors and dielectrics. Surface current densities may exist at the surface of conductors and in particular perfect conductors. Thus, in many practical applications, we do not need to worry about charge or current densities at the interface. In particular, two types of interfaces are unique and often useful:

(1) Interfaces between perfect dielectrics (lossless dielectrics).
(2) Interfaces between a perfect dielectric and a perfect conductor.

In the first of these, there are neither current densities nor charge densities at the interface. The interface conditions therefore reduce to those in **Table 11.4**.

Table 11.4 Summary of interface conditions between two perfect dielectrics

	Electric field		Magnetic field	
Tangential components	$E_{1t} = E_{2t}$	$D_{1t}/\varepsilon_1 = D_{2t}/\varepsilon_2$	$H_{1t} = H_{2t}$	$B_{1t}/\mu_1 = B_{2t}/\mu_2$
Normal components	$D_{1n} = D_{2n}$	$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n}$	$B_{1n} = B_{2n}$	$\mu_1 H_{1n} = \mu_2 H_{2n}$

The second type of interface discussed here is that between a perfect dielectric and a perfect conductor. In this case, the overriding condition is that of the conductor, that is, that all fields in the perfect conductor must be zero. Assuming material (2) is the perfect conductor, E_{2t}, H_{2t}, D_{2n}, and B_{2n} are zero. The interface conditions are given in **Table 11.5**.

Table 11.5 Summary of interface conditions between a perfect dielectric and a perfect conductor

	Electric field		Magnetic field	
Tangential components	$E_{1t} = E_{2t} = 0$	$D_{1t} = D_{2t} = 0$	$H_{1t} = J_s^*$	$B_{1t} = \mu_1 J_s^*$
Normal components	$D_{1n} = \rho_s$	$E_{1n} = \rho_s/\varepsilon_1$	$B_{1n} = 0$	$H_{1n} = 0$

*The directions of J_s and H_{1t} or B_{1t} are related through the right-hand rule

Example 11.5 Interface Conditions for Electromagnetic Fields A magnetic field exists in material (1) in **Figure 11.3a** as $\mathbf{H}_1(x,y,z,t) = (\hat{\mathbf{x}} + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}3)\cos 377t$ [A/m]. Material (1) has a relative permeability of 100. The interface between material (1) and free space [material (2)] is on the plane $z = 0$ and there are no currents on the interface. Calculate the magnetic field intensity **H** and the magnetic flux density **B** in material (2).

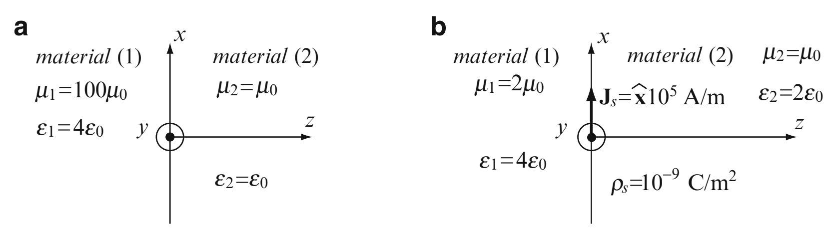

Figure 11.3 (**a**) Interface between two materials. (**b**) Interface between two general materials with a current density on the interface

Solution: The magnetic field intensity has a tangential component and a normal component. In vector components, these are

$$\mathbf{H}_{1t}(x,y,z,t) = (\hat{\mathbf{x}} + \hat{\mathbf{y}}2)\cos 377t \quad \mathrm{H}_{1n}(x,y,z,t) = -\hat{\mathbf{z}}3\cos 377t \quad [\mathrm{A/m}]$$

The tangential components of the magnetic field intensity are continuous across the interface (no current density on the interface):

$$\mathbf{H}_{2t} = \mathbf{H}_{1t} = (\hat{\mathbf{x}} + \hat{\mathbf{y}}2)\cos 377t \quad [\mathrm{A/m}]$$

The normal component of the magnetic flux density is continuous across the interface:

$$\mu_1 \mathbf{H}_{1n} = \mu_2 \mathbf{H}_{2n} \rightarrow \mathbf{H}_{2n} = \frac{\mu_1}{\mu_2}\mathbf{H}_{1n} = \frac{\mu_{r1}\mu_0}{\mu_0}\mathbf{H}_{1n} = \mu_{r1}\mathbf{H}_{1n} = -\hat{\mathbf{z}}300\cos 377t$$

Thus, the magnetic field intensity in material (2) is

$$\mathbf{H}_2 = \mathbf{H}_{2t} + \mathbf{H}_{2n} = (\hat{\mathbf{x}} + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}300)\cos 377t \quad \left[\frac{\mathrm{A}}{\mathrm{m}}\right]$$

The magnetic flux density in material (2) is

$$\mathbf{B}_2 = \mu_0 \mathbf{H}_2 = (\hat{\mathbf{x}} + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}300)\mu_0\cos 377t \quad [\mathrm{T}]$$

Exercise 11.2 The configuration in **Example 11.5** is given. The electric field intensity in material (1) is $\mathbf{E}_1(x,y,z,t) = k(\hat{\mathbf{x}} + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}3)\cos 377t$ [V/m], where k is a constant. Calculate the electric field intensity and electric flux density in material (2). Assume there are no charges on the interface.

Answer

$$\mathbf{E}_2(x,y,z,t) = k(\hat{\mathbf{x}} + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}12)\cos 377t \quad [\mathrm{V/m}],$$
$$\mathbf{D}_2(x,y,z,t) = k(\hat{\mathbf{x}} + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}3)\varepsilon_0\cos 377t \quad [\mathrm{C/m^2}]$$

Example 11.6 Interface Conditions for the Static Electric and Magnetic Fields An interface between two general materials contains both a current density given as $\mathbf{J}_s = \hat{\mathbf{x}}10^5$ A/m and a uniform surface charge density given as $\rho_s = 10^{-9}$ C/m^2. The static magnetic field intensity and static electric field intensity in material (1) are

$$\mathbf{H}_1 = \hat{\mathbf{x}}10^5 + \hat{\mathbf{y}}10^5 - \hat{\mathbf{z}}10^5 \quad [\mathrm{A/m}], \qquad \mathbf{E}_1 = \hat{\mathbf{x}}100 + \hat{\mathbf{y}}20 - \hat{\mathbf{z}}100 \quad [\mathrm{V/m}]$$

For the material properties given in **Figure 11.3b**, ($\mu_1 = 2\mu_0$, $\mu_2 = \mu_0$ [H/m], $\varepsilon_1 = 4\varepsilon_0$, and $\varepsilon_2 = 2\varepsilon_0$ [F/m]), find:

(a) The electric field intensity in material (2).
(b) The magnetic flux density in material (2).

Note: Static electric and magnetic fields are independent of each other.

Solution: Since both current densities and charge densities exist on the interface, we must use the general interface conditions in **Table 11.2**:

(a) The tangential and normal vector components of **E** in material (1) are

$$\mathbf{E}_{1t} = \hat{\mathbf{x}}100 + \hat{\mathbf{y}}20, \quad \mathbf{E}_{1n} = -\hat{\mathbf{z}}100 \quad [\mathrm{V/m}]$$

The tangential component of the electric field intensity is continuous across the interface:

$$\mathbf{E}_{2t} = \mathbf{E}_{1t} = \hat{\mathbf{x}}100 + \hat{\mathbf{y}}20 \quad [\mathrm{V/m}]$$

The normal component of the electric field intensity is discontinuous across the interface:

$$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s \quad \rightarrow \quad E_{2n} = \frac{\varepsilon_1 E_{1n} - \rho_s}{\varepsilon_2} \quad [\mathrm{V/m}]$$

where we assume E_{1n} points away from the interface and E_{2n} points toward the interface. This gives

$$E_{2n} = \frac{4\varepsilon_0(-100) - 10^{-9}}{2\varepsilon_0} = -200 - \frac{10^{-9}}{2 \times 8.854 \times 10^{-12}} = -256.47 \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]$$

Thus, the electric field intensity in material (2) is

$$\mathbf{E}_2 = \mathbf{E}_{2t} + \mathbf{E}_{2n} = \hat{\mathbf{x}}100 + \hat{\mathbf{y}}20 - \hat{\mathbf{z}}256.47 \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right]$$

(b) First, we write the magnetic flux density (**B** = μ**H**)

$$\mathbf{B}_1 = \hat{\mathbf{x}}2\mu_0 \times 10^5 + \hat{\mathbf{y}}2\mu_0 \times 10^5 - \hat{\mathbf{z}}2\mu_0 \times 10^5 \quad [\mathrm{T}]$$

For convenience we separate the magnetic flux density into its tangential and normal components as follows:

$$\mathbf{B}_{1t} = \mu_1\mathbf{H}_{1t} = \hat{\mathbf{x}}2\mu_0 \times 10^5 + \hat{\mathbf{y}}2\mu_0 \times 10^5, \quad \mathbf{B}_{1n} = \mu_1\mathbf{H}_{1n} = -\hat{\mathbf{z}}2\mu_0 \times 10^5 \quad [\mathrm{T}]$$

The tangential component of the magnetic flux density is discontinuous across the interface (**Table 11.3**):

$$\hat{\mathbf{n}} \times \left(\frac{\mathbf{B}_1}{\mu_1} - \frac{\mathbf{B}_2}{\mu_2} \right) = \mathbf{J}_s \quad \rightarrow \quad \hat{\mathbf{n}} \times \mathbf{B}_2 = -\mathbf{B}_{2t} = \mu_2 \left(\frac{\hat{\mathbf{n}} \times \mathbf{B}_1}{\mu_1} - \mathbf{J}_s \right) \quad [\mathrm{T}]$$

Since the normal must point into medium (1), we write $\hat{\mathbf{n}} = -\hat{\mathbf{z}}$ and get

$$-\hat{\mathbf{z}} \times \mathbf{B}_2 = \mathbf{B}_{2t} = \mu_2 \left[\frac{-\hat{\mathbf{z}} \times \mathbf{B}_1}{\mu_1} - \mathbf{J}_s \right] \quad [\mathrm{T}]$$

or

$$\mathbf{B}_{2t} = \mu_2 \left(\frac{\hat{\mathbf{z}} \times \mathbf{B}_1}{\mu_1} + \mathbf{J}_s \right) \quad [\mathrm{T}]$$

Substituting for $\mathbf{B}_1$ and $\mathbf{J}_s$,

$$\mathbf{B}_{2t} = \mu_0 \left(\frac{\hat{\mathbf{z}} \times \left(\hat{\mathbf{x}} 2\mu_0 \times 10^5 + \hat{\mathbf{y}} 2\mu_0 \times 10^5 - \hat{\mathbf{z}} 2\mu_0 \times 10^5 \right)}{2\mu_0} + \hat{\mathbf{x}} \times 10^5 \right) = \hat{\mathbf{y}} 10^5 \mu_0 \quad [\mathrm{T}]$$

The normal component of $\mathbf{B}$ is continuous across the interface:

$$\mathbf{B}_{2n} = \mathbf{B}_{1n} = -\hat{\mathbf{z}} 2 \times 10^5 \mu_0 \quad [\mathrm{T}]$$

Thus, the magnetic flux density in material (2) is

$$\mathbf{B}_2 = \hat{\mathbf{y}} 10^5 \mu_0 - \hat{\mathbf{z}} 2 \times 10^5 \mu_0 \quad [\mathrm{T}]$$

11.5 Particular Forms of Maxwell's Equations

Maxwell's equations as given in **Section 11.2.2** are general and apply to all electromagnetic situations and for any type of time dependency. In this sense, whenever there is a need to solve an electromagnetic problem, we can start with **Eqs. (11.24)** through (**11.27**) or, if integral representation is more convenient, with **Eqs. (11.28)** through (**11.31**). However, more often than not, there is no need to resort to the general system. For example, we might need to solve the equations at low frequencies, in which case the displacement currents might be negligible or do not exist. In still other situations the current densities or charge densities in the system, or both, are negligible. Some of these representations are particularly useful, and, therefore, we discuss these here, before we apply them to particular electromagnetic problems in the following chapters. In particular, the time-harmonic representation of the equations is often useful.

11.5.1 Time-Harmonic Representation

In a time-harmonic field, the time dependency is sinusoidal. This is a form often encountered in engineering and, as is well known from circuit theory, offers distinct advantages in analysis. As with circuits, the time-harmonic form can be used for almost any waveform through the use of Fourier series. This approach implies linearity in relations. The same is true in electromagnetics. Much of the remaining material in this book will be based on the time-harmonic representation of the electromagnetic field equations. For this reason, we present now Maxwell's equations in time-harmonic form.

There are two basic differences between time-dependent and time-harmonic forms:

(1) Field variables as well as sources are phasors.
(2) The time derivative operator d/dt is replaced by $j\omega$.

Before discussing these any further, it is useful to review the concept of phasors, particularly because we will use phasors in conjunction with vector variables.

The phasor notation is a method of representing complex numbers. Consider a complex number $b = u_0 + jv_0$, where $j = \sqrt{-1}$. The complex number b can be represented in a plane, called a ***complex plane***, as in **Figure 11.4a**. The real part of b is u_0, and it is the projection of b on the real axis, whereas the imaginary part of a is v_o and represents its projection on the imaginary axis.

Instead of writing b in the above form, we can also write b in terms of a magnitude and an angle. The phasor notation is based on the latter form and arises from Euler's equation:

$$be^{j\varphi} = b\cos(\varphi) + jb\sin(\varphi) \tag{11.58}$$

That is, if the radius of a circle of magnitude b makes an angle φ with the real axis, its projections on the real and imaginary axes are $b\cos(\varphi)$ and $b\sin(\varphi)$, respectively. These concepts are shown in **Figure 11.4b**. The phase angle φ can be general and we will assume that it has the form $\varphi = \omega t + \theta$, where ω is an angular frequency, t is time, and θ is some fixed phase angle. Substituting this in **Eq. (11.58)** gives

$$be^{j\varphi} = be^{j(\omega t+\theta)} = be^{j\omega t}e^{j\theta} = b\cos(\omega t + \theta) + jb\sin(\omega t + \theta) \tag{11.59}$$

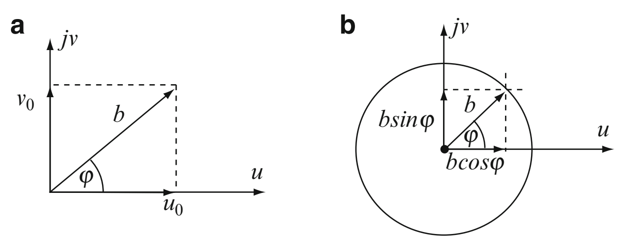

Figure 11.4 (**a**) Representation of a complex number b. (**b**) Harmonic representation of a general complex number of magnitude b

A sinusoidal function of the type often used in fields is

$$f_1(x, y, z, t) = A_0(x, y, z)\cos(\omega t + \theta) \quad \text{or} \quad f_2(x, y, z, t) = A_0(x, y, z)\sin(\omega t + \theta) \tag{11.60}$$

The phasor notation now allows us to write

$$A_0(x, y, z)\cos(\omega t + \theta) = \text{Re}\{A_0(x, y, z)e^{j\omega t}e^{j\theta}\} \tag{11.61}$$

$$A_0(x, y, z)\sin(\omega t + \theta) = \text{Im}\{A_0(x, y, z)e^{j\omega t}e^{j\theta}\} \tag{11.62}$$

where A_0 is real and independent of time, Re{ } means the real part of the function, and Im{ } means the imaginary part of the function. Finally, we define the ***phasor*** as that part of the function which does not contain time; that is,

$$A_p(x, y, z) = A_0(x, y, z)e^{j\theta} \tag{11.63}$$

This is sometimes written as an amplitude and phase as

$$A_p(x, y, z) = A_0(x, y, z)\angle\theta \tag{11.64}$$

Summarizing, the phasor can be written in three different forms:

$$A_p(x, y, z) = A_0(x, y, z)e^{j\theta} = A_0(x, y, z)\angle\theta = A_0(x, y, z)\cos\theta + jA_0(x, y, z)\sin\theta \tag{11.65}$$

The first form is called the exponential form, the second is the polar form, and the third is the rectangular form. Most of our work in this book will be carried out in the exponential form. On occasion, we will use the polar form, particularly for presentation of results because it is a more compact method. The rectangular form is also convenient in some cases because of its explicit representation in complex variables. Using the exponential form of the phasor, the time domain form, $A(x,y,z,t)$, can be written as

$$A(x,y,z,t) = \mathrm{Re}\left\{A_0(x,y,z)e^{j\theta}e^{j\omega t}\right\} \quad (11.66)$$

Now, the reason for the use of phasors is apparent: It allows representation of fields in terms of a magnitude (A_0) and a phase angle (θ) without explicitly considering time, since the phasor does not contain the term $e^{j\omega t}$. When we need to convert phasors to time, the term $e^{j\omega t}$ is included as in **Eq. (11.66)**.

In the above discussion, we assumed A to be a scalar function. However, the definition of the phasor, because it has to do with any complex number, applies equally well to vectors. All we need to do is replace the scalar A by a vector **A**. Similarly, the amplitude A_0 now becomes a vector $\mathbf{A}_0$. The phasor form of **A** is $\mathbf{A}_p$. Thus, given a time-dependent vector $\mathbf{A}(x,y,z,t)$, the phasor form of $\mathbf{A}(x,y,z,t)$ is $\mathbf{A}_p(x,y,z)$, and given the phasor form $\mathbf{A}_p$, the time-dependent vector is $\mathbf{A}(x,y,z,t) = \mathrm{Re}\{\mathbf{A}_p(x,y,z)e^{j\omega t}\}$.

One of the most distinct advantages in working with phasors is the ease with which time derivatives are performed. The time derivative of a general vector $\mathbf{A}(x,y,z,t)$ is

$$\frac{d}{dt}(\mathbf{A}(x,y,z,t)) = \mathrm{Re}\left\{j\omega\mathbf{A}_p(x,y,z)e^{j\omega t}\right\} \quad (11.67)$$

In the practical use of phasors, we do not keep the term $e^{j\omega t}$, but it is understood to exist. Neither do we denote the phasor in any other way. In this section, the phasor was denoted with a subscript p. In later use we will drop this notation because it will normally be understood from the context if we are using phasors or not.

Example 11.7 Phasors A time-dependent electric field intensity is given as $\mathbf{E} = \hat{\mathbf{x}}(10\pi + j20\pi)\cos\left(10^6t - 120y\right)$ [V/m]. Write the electric field intensity as a phasor using the following:

(a) The rectangular notation.
(b) The polar representation.
(c) The exponential representation.

Solution:

(a) First, we must write the electric field intensity as follows:

$$\mathbf{E} = \hat{\mathbf{x}}\left(10\pi\cos\left(10^6t - 120y\right) + j20\pi\cos\left(10^6t - 120y\right)\right)$$
$$= \hat{\mathbf{x}}\left(10\pi\cos\left(10^6t - 120y\right) + 20\pi\cos\left(10^6t - 120y + \pi/2\right)\right) \quad [\mathrm{V/m}]$$

Each term can be written in rectangular form noting its amplitude and phase. Comparison with **Eq. (11.59)** shows that $\omega t = 10^6t$, $\theta_1 = -120y$, and $\theta_2 = -120y + \pi/2$. Removing the term $e^{j\omega t}$, the phasor form becomes [see **Eq. (11.65)**]

$$\mathbf{E} = \hat{\mathbf{x}}\left(10\pi\cos(-120y) + j10\pi\sin(-120y) + 20\pi\cos(-120y + \pi/2) + j20\pi\sin(-120y + \pi/2)\right) \quad [\mathrm{V/m}]$$

Or, writing $\cos(-120y) = \cos(120y)$, $\sin(-120y) = -\sin(120y)$, $\cos(-120y + \pi/2) = \sin(120y)$, and $\sin(-120y + \pi/2) = \cos(120y)$, we can simplify the expression

$$\mathbf{E} = \hat{\mathbf{x}}[10\pi\cos120y - j10\pi\sin120y + 20\pi\sin120y + j20\pi\cos120y]$$
$$= \hat{\mathbf{x}}\left[(10\pi + 20\pi)\cos120y - j(10\pi - 20\pi)\sin120y\right] \quad [\mathrm{V/m}].$$

(b) In polar representation, we get

$$\mathbf{E} = \hat{\mathbf{x}}10\pi\,\underline{/-120y} + \hat{\mathbf{x}}20\pi\,\underline{/-120y + \pi/2} \quad [\mathrm{V/m}].$$

(c) In exponential form,

$$\mathbf{E} = \hat{\mathbf{x}}\left(10\pi e^{-j120y} + 20\pi e^{-j120y} e^{j\pi/2}\right) \quad [\mathrm{V/m}].$$

Example 11.8 Phasors The magnetic field intensity in a material is given as a phasor:

$$\mathbf{H} = [\hat{\mathbf{x}}(100 + j50) + \hat{\mathbf{y}}50 + \hat{\mathbf{z}}100]e^{j60}e^{-j3x} \quad [\mathrm{A/m}]$$

(a) Write the magnetic field intensity in rectangular form.
(b) What is the time-dependent magnetic field intensity **H** in the material?

Solution:

(a) First, we write the magnetic field intensity as

$$\mathbf{H} = \hat{\mathbf{x}}\left(100e^{j60}e^{-j3x} + 50e^{j60}e^{-j3x}e^{j\pi/2}\right) + \hat{\mathbf{y}}50e^{j60}e^{-j3x} + \hat{\mathbf{z}}100e^{j60}e^{-j3x} \quad [\mathrm{A/m}]$$

The magnetic field intensity in rectangular form is

$$\begin{aligned}\mathbf{H} = \hat{\mathbf{x}}&\left[100\cos(60-3x) + j100\sin(60-3x) + 50\cos(60-3x+\pi/2) + j50\sin(60-3x+\pi/2)\right] \\ &+ \hat{\mathbf{y}}\left[50\cos(60-3x) + j50\sin(60-3x)\right] + \hat{\mathbf{z}}\left[100\cos(60-3x) + j100\sin(60-3x)\right] \quad [\mathrm{A/m}].\end{aligned}$$

(b) The time-dependent field is written as

$$\begin{aligned}\mathbf{H}(t) = \mathrm{Re}\left\{\mathbf{H}e^{j\omega t}\right\} &= \hat{\mathbf{x}}\left[100\cos(\omega t + 60 - 3x) + 50\cos(\omega t + 60 - 3x + \pi/2)\right] \\ &+ \hat{\mathbf{y}}50\cos(\omega t + 60 - 3x) + \hat{\mathbf{z}}100\cos(\omega t + 60 - 3x) \qquad [\mathrm{A/m}].\end{aligned}$$

11.5.2 Maxwell's Equations: The Time-Harmonic Form

With the notation given above, we can now write Maxwell's equations in terms of phasors. Implicit in this development is linearity of material properties. Assuming that all vector and scalar quantities are phasors, we simply replace d/dt or $\partial/\partial t$ by $j\omega$ in **Eqs. (11.24)** through **(11.34)**. The time-harmonic differential and integral forms of Maxwell's equations together with the constitutive relations and the Lorentz force are summarized in **Table 11.6**.

Table 11.6 Summary of the time-harmonic electromagnetic field equations

	Differential form		Integral form	
Maxwell's equations	$\nabla \times \mathbf{E} = -j\omega\mathbf{B}$	(11.68)	$\oint_C \mathbf{E}\cdot d\mathbf{l} = -j\omega\int_s \mathbf{B}\cdot d\mathbf{s}$ [V]	(11.72)
	$\nabla \times \mathbf{H} = J + j\omega\mathbf{D}$ [A/m^2]	(11.69)	$\oint_C \mathbf{H}\cdot d\mathbf{l} = \int_s (\mathbf{J} + j\omega\mathbf{D})\cdot d\mathbf{s}$ [A]	(11.73)
	$\nabla \cdot \mathbf{D} = \rho_v$ [C/m^3]	(11.70)	$\oint_s \mathbf{D}\cdot d\mathbf{s} = Q$ [C]	(11.74)
	$\nabla \cdot \mathbf{B} = 0$	(11.71)	$\oint_s \mathbf{B}\cdot d\mathbf{s} = 0$	(11.75)
Constitutive relations			$\mathbf{B} = \mu\mathbf{H}$ [T]	(11.76)
			$\mathbf{D} = \varepsilon\mathbf{E}$ [C/m^2]	(11.77)
The Lorentz force equation			$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$ [N]	(11.78)

Note that the constitutive relations and the Lorentz force equations have not changed although all vector quantities are now assumed to be phasors. Of course, velocity is still a real number. ε and μ remain unaffected by the phasor notation. The charge Q or the charge density ρ_v may, in some cases, be time dependent, in which case they also become phasors.

Another important point to be noted here is that if displacement currents in **Eq. (11.69)** or **(11.73)** are neglected, the pre-Maxwell system of equations is obtained, but the fields are now time-harmonic fields. This system of equations, which is characterized by slow varying fields (and hence the neglection of displacement currents), is called the ***quasi-static field equations***. The term quasi-static means that the equations are static-like in the sense that the equations satisfy Laplace's or Poisson's equations. One of the advantages of this form is that it extends many of the properties as well as the methods used for static fields to time-dependent fields.

Finally, if we set all time derivatives to zero, the purely static equations used for electrostatics and magnetostatics are obtained. As was said previously, under this condition, the electric field **Eqs. (11.68)** and **(11.70)** and the magnetic field **Eqs. (11.69)** and **(11.71)** (or their integral counterparts) are decoupled, and there is no need to discuss them as a system of equations.

11.5.3 Source-Free Equations

The general forms of Maxwell's equations can sometimes be simplified if the sources do not need to be taken into account. Under these conditions, the current density **J**, the charge density ρ_v, or both are removed from the equations and a much simpler form of the equations is obtained. This is true in the time-dependent or phasor forms of the equations. The time-dependent and time-harmonic source-free Maxwell's equations are summarized in **Tables 11.7** and **11.8**.

Table 11.7 The source-free time-dependent Maxwell's equations

	Differential	Integral
Faraday's law	$\nabla\times\mathbf{E}=-\dfrac{\partial\mathbf{B}}{\partial t}$	$\oint_C\mathbf{E}\cdot d\mathbf{l}=-\dfrac{d\Phi}{dt}$ [V]
Ampere's law	$\nabla\times\mathbf{H}=-\dfrac{\partial\mathbf{D}}{\partial t}$ [A/m^2]	$\oint_C\mathbf{H}\cdot d\mathbf{l}=\int_s\dfrac{\partial\mathbf{D}}{dt}\cdot d\mathbf{s}$ [A]
Gauss's law	$\nabla\cdot\mathbf{D}=0$	$\oint_s\mathbf{D}\cdot d\mathbf{s}=0$
No monopoles	$\nabla\cdot\mathbf{B}=0$	$\oint_s\mathbf{B}\cdot d\mathbf{s}=0$

Table 11.8 The source-free time-harmonic Maxwell's equations

Faraday's law	$\nabla\times\mathbf{E}=-j\omega\mathbf{B}$	$\oint_C\mathbf{E}\cdot d\mathbf{l}=-j\omega\oint_s\mathbf{B}\cdot d\mathbf{s}$ [V]
Ampere's law	$\nabla\times\mathbf{H}=j\omega\mathbf{D}$ [A/m^2]	$\oint_C\mathbf{H}\cdot d\mathbf{l}=j\omega\int_s\mathbf{D}\cdot d\mathbf{s}$ [A]
Gauss's law	$\nabla\cdot\mathbf{D}=0$	$\oint_s\mathbf{D}\cdot d\mathbf{s}=0$
No monopoles	$\nabla\cdot\mathbf{B}=0$	$\oint_s\mathbf{B}\cdot d\mathbf{s}=0$

That these equations are simpler than those given in **Eqs. (11.24)** through **(11.27)** or **(11.28)** through **(11.31)** is obvious. For example, the divergence of **D** (or **E**) is zero, which makes the electric field solenoidal. The fact that we do not need to treat sources makes the solution of field problems much simpler, provided that the conditions under which these equations apply are satisfied.

We will not expand on this here except to point out that treatment of fields under source-free conditions is quite common. You may wish to think about it in this fashion: If you are interested in evaluating the field distribution in a volume such as a room due to say, terrestrial magnetism, or the transmission from a distant TV station, there is little choice but to solve the problem in the absence of sources. In both of these cases, we have no knowledge of the sources, and, in fact, we may not even know where the sources are located. The fields, however, are real. We can measure them at various locations, we can find their distribution in space, and we can calculate a number of other properties related to the fields.

11.6 Summary

The main topic in this chapter is the introduction of displacement current density in Ampere's law and its consequences. The final result is Maxwell's equations, which include the postulates in the previous chapters but also the modification due to displacement currents. The displacement current density modifies Ampere's law by adding the term $\mathbf{J}_d = \partial\mathbf{D}/\partial t$ [A/m^2] as follows:

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \left[\frac{\mathrm{A}}{\mathrm{m}^2}\right] \tag{11.6}$$

This, together with Faraday's and Gauss's laws, forms what are called ***Maxwell's equations*** given below in differential form (left) and integral form (right):

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (11.24) \quad \text{or :} \quad \oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial \Phi}{\partial t} \quad [\mathrm{V}] \quad (11.28)$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad [\mathrm{A/m^2}] \quad (11.25) \quad \text{or :} \quad \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_s \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{s} \quad [\mathrm{A}] \quad (11.29)$$

$$\nabla \cdot \mathbf{D} = \rho_v \quad [\mathrm{C/m^3}] \quad (11.26) \quad \text{or :} \quad \oint_s \mathbf{D} \cdot d\mathbf{s} = Q \quad [\mathrm{C}] \quad (11.30)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (11.27) \quad \text{or :} \quad \oint_s \mathbf{B} \cdot d\mathbf{s} = 0 \quad (11.31)$$

The material constitutive relations $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$ and the Lorentz force $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ are part of the general system of equations called the Maxwell–Lorentz equations (see **Table 11.1**). The third constitutive relation, $\mathbf{J} = \sigma\mathbf{E}$, applies in conducting media.

Time-dependent potentials are defined based on the properties of the curl and divergence of fields:

$$\mathbf{E} = -\nabla V, \quad \text{if} \quad \nabla \times \mathbf{E} = 0 \tag{11.38}$$

V is the electric scalar potential (voltage).

$$\mathbf{H} = -\nabla \psi, \quad \text{if} \quad \nabla \times \mathbf{H} = 0 \tag{11.39}$$

ψ is the magnetic scalar potential.

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \text{because} \quad \nabla \cdot \mathbf{B} = 0 \tag{11.40}$$

A is the magnetic vector potential.

The time-dependent electric field intensity, based on Ampere's law **[Eq. (11.24)]**, is

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla V \quad \left[\frac{\mathrm{V}}{\mathrm{m}}\right] \tag{11.45}$$

Gauges define the divergence of vector potentials (in this case the magnetic vector potential).
$\nabla \cdot \mathbf{A} = 0$ for static fields (Coulomb's gauge) and

$$\nabla \cdot \mathbf{A} = -\mu\varepsilon \frac{\partial V}{\partial t} \tag{11.49}$$

for time-dependent fields (Lorenz's gauge).

Interface conditions for time-dependent fields are identical to those for static fields as discussed in **Chapters 4** and **9**. These are summarized in **Tables 11.2** through **11.5** (see **Figure 11.2** for reference):

$$E_{1t} = E_{2t}, \quad \frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2} \quad \text{and} \quad D_{1n} - D_{2n} = \rho_s, \quad \varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$$

$$\hat{\mathbf{n}} \times (\mathbf{H}_{1t} - \mathbf{H}_{2t}) = \mathbf{J}_s, \quad \hat{\mathbf{n}} \times \left(\frac{\mathbf{B}_{1t}}{\mu_1} - \frac{\mathbf{B}_{2t}}{\mu_2} \right) = \mathbf{J}_s \quad \text{and} \quad B_{1n} = B_{2n}, \quad \mu_1 H_{1n} = \mu_2 H_{2n}$$

Electromagnetic fields are often represented in terms of phasors. ***Phasor representation*** of any function A (scalar or vector) is as follows:

$$A_p(x,y,z) = A_0(x,y,z)e^{j\theta} = A_0(x,y,z)\angle\theta = A_0(x,y,z)\cos\theta + jA_0(x,y,z)\sin\theta \tag{11.65}$$

Transformation into the time domain is as follows:

$$A(x,y,z,t) = \text{Re}\left\{A_0(x,y,z)e^{j\theta}e^{j\omega t}\right\} \tag{11.66}$$

$$\frac{d}{dt}(A(x,y,z,t)) = \text{Re}\left\{j\omega A_p(x,y,z)e^{j\omega t}\right\} \tag{11.67}$$

Time-harmonic field equations play an important role in electromagnetics. Maxwell's equations in the frequency domain (see **Table 11.6**) are:

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B} \quad (11.68) \quad \text{or}: \quad \oint_C \mathbf{E} \cdot d\mathbf{l} = -j\omega \int_s \mathbf{B} \cdot d\mathbf{s} \quad [\text{V}] \tag{11.72}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega\mathbf{D} \quad [\text{A/m}^2] \quad (11.69) \quad \text{or}: \quad \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_s (\mathbf{J} + j\omega\mathbf{D}) \cdot d\mathbf{s} \quad [\text{A}] \tag{11.73}$$

$$\nabla \cdot \mathbf{D} = \rho_v \quad [\text{C/m}^3] \quad (11.70) \quad \text{or}: \quad \oint_s \mathbf{D} \cdot d\mathbf{s} = Q \quad [\text{C}] \tag{11.74}$$

$$\nabla \cdot \mathbf{B} = 0 \quad (11.71) \quad \text{or}: \quad \oint_s \mathbf{B} \cdot d\mathbf{s} = 0 \tag{11.75}$$

where **E, H, D, B**, and **J** are vector phasors and Q and ρ_v are scalar phasors. Note however that we do not mark these in any particular way—it is understood from the context when these quantities must be phasors.

Source-free equations are obtained by setting $\mathbf{J} = 0$, $\rho_v = 0$ in either the time or frequency domain equations. These are summarized in **Tables 11.7** and **11.8**.

Problems

Maxwell's Equations, Displacement Current, and Continuity

11.1 Displacement Current Density. A magnetic flux density, $\mathbf{B} = \hat{\mathbf{y}}0.1(\cos 100t)(\cos 5z)$ [T] exists in a linear, isotropic, homogeneous material characterized by ε and μ. Find the displacement current density in the material if there are no source charges or current densities in the material.

11.2 Displacement Current in Spherical Capacitor. Determine the displacement current I_d [A] which flows between two concentric, conducting spherical shells of radii a and b [m] where $b > a$ in free space with a voltage difference $V_0\sin\omega t$ [V] applied between the spheres.

11.3 Application: Displacement Current in Cylindrical Capacitor. A voltage source $V_0\sin\omega t$ [V] is connected between two concentric conductive cylinders $r = a$ and $r = b$ [A], where $b > a$, with length L [A]. $\varepsilon = \varepsilon_r\varepsilon_0$ [F/m], $\mu = \mu_0$ [H/m], and $\sigma = 0$ for $a < r < b$. Neglect any end effects and find:

(a) The displacement current density at any point $a < r < b$.
(b) The total displacement current I_d flowing between the two cylinders.

11.4 Conservation of Charge and Displacement Current. Show that the displacement current in Maxwell's second equation (Ampere's law) is a direct consequence of the law of conservation of charge.

11.5 Application: Displacement and Conduction Current Densities in Lossy Capacitor. A lossy dielectric is located between two parallel plates which are connected to an AC source (**Figure 11.5**). Material properties of the dielectric are $\varepsilon = 9\varepsilon_0$ [F/m], $\mu = \mu_0$ [H/m], and $\sigma = 4$ S/m. The source is given as $V = 1\cos\omega t$ [V]. Calculate the frequency at which the magnitude of the displacement current density is equal to the magnitude of the conduction current density. Assume all material properties are independent of frequency.

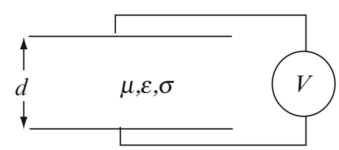

Figure 11.5

11.6 Displacement and Conduction Current Densities. A capacitor is made of two parallel plates with a dielectric between them. The relative permittivity of the dielectric is $\varepsilon_r = 4$, the distance between plates is $d = 1$ mm, and the area of each plate is $S = 100\ \text{mm}^2$. Because of an accident, the dielectric became wet with a salt solution and therefore became conducting with conductivity $\sigma = 10^{-3}$ S/m. If the capacitor is connected to an AC source of amplitude V [V] and frequency ω [rad/s], show that:

(a) The ratio between the amplitudes of the conduction current density and displacement current density is

$$\frac{J_{cond}}{J_{disp}} = \frac{\sigma}{\omega\varepsilon}.$$

(b) The conduction and displacement current densities are 90° out of phase.

11.7 Application: Lossy Capacitor. The capacitor in **Problem 11.6** is connected to a 12 V DC source and charged for a long period of time. Now the source is disconnected. Find the time constant of discharge of the capacitor.

11.8. Displacement Current. An AC generator operating at a frequency of 1 GHz is connected with a wire to a small conducting sphere of radius $a = 10$ mm at some distance away (see **Figure 11.6**). If the sphere is in free space, calculate the current in the wire. Neglect any effect the ground may have. The generator generates a sinusoidal voltage of amplitude 100 V.

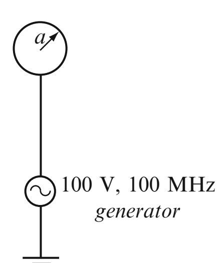

Figure 11.6

Maxwell's Equations

11.9 Maxwell's Equations. What value of A and β are required if the two fields:

$$\mathbf{E} = \hat{\mathbf{y}}120\pi\cos\left(10^6\pi t - \beta x\right) \quad [\mathrm{V/m}]$$
$$\mathbf{H} = \hat{\mathbf{z}}A\pi\cos\left(10^6\pi t - \beta x\right) \quad [\mathrm{A/m}]$$

satisfy Maxwell's equations in a linear, isotropic, homogeneous medium with $\varepsilon_r = \mu_r = 4$ and $\sigma = 0$? Assume there are no current or charge densities in space.

11.10 Dependency in Maxwell's Equations. Show that **Eq. (11.8)** ($\nabla \cdot \mathbf{B} = 0$) can be derived from **Eq. (11.5)** and, therefore, is not an independent equation.

11.11 Dependency in Maxwell's Equations. Show that **Eq. (11.7)** ($\nabla \cdot \mathbf{D} = \rho_v$) can be derived from **Eq. (11.6)** with the use of the continuity equation [**Eq. (11.13)**] and, therefore, is not an independent equation.

11.12 The Lorenz Condition (Gauge). Show that the Lorenz condition in **Eq. (11.49)** leads to the continuity equation. **Hint**: Use the expression for electric potential due to a general volume charge distribution and the expression for the magnetic vector potential due to a general current density in a volume.

11.13 Maxwell's Equations. Maxwell's equations in **Eqs. (11.24)** through **(11.27)** are equivalent to eight scalar equations. Find these equations by writing the vector fields explicitly in Cartesian coordinates and equating components.

11.14 Maxwell's Equations in Cylindrical Coordinates. Write Maxwell's equations explicitly in cylindrical coordinates by expanding the expressions in **Eqs. (11.24)** through **(11.27)**.

11.15 Maxwell's Equations. A time-dependent magnetic field is given as $\mathbf{B} = \hat{\mathbf{x}}20e^{j\left(10^4 t + 10^{-4} z\right)}$ [T] in a material with properties $\varepsilon_r = 9$ and $\mu_r = 1$. Assume there are no sources in the material. Using Maxwell's equations:

(a) Calculate the electric field intensity in the material.
(b) Calculate the electric flux density and the magnetic field intensity in the material.

11.16 Maxwell's Equations. A time-dependent electric field intensity is given as $\mathbf{E} = \hat{\mathbf{x}}10\pi\cos\left(10^6 t - 50z\right)$ [V/m]. The field exists in a material with properties $\varepsilon_r = 4$ and $\mu_r = 1$. Given that $\mathbf{J} = 0$ and $\rho_v = 0$, calculate the magnetic field intensity and magnetic flux density in the material.

Potential Functions

11.17 Current Density as a Primary Variable in Maxwell's Equations. Given: Maxwell's equations in a linear, isotropic, homogeneous medium. Assume that there are no source current densities and no charge densities anywhere in the solution space. An induced current density $\mathbf{J}_e$ [A/m^2] exists in conducting materials. Assume the whole space is conducting, with a very low conductivity, σ [S/m]. Rewrite Maxwell's equations in terms of the current density $\mathbf{J}_e = \sigma\mathbf{E}$. In other words, assume you need to solve for $\mathbf{J}_e$ directly.

11.18 Magnetic Scalar Potential. Write an equation, equivalent to Maxwell's equations in terms of a magnetic scalar potential in a linear, isotropic, homogeneous medium. State the conditions under which this can be done:

(a) Show that Maxwell's equations reduce to a second-order partial differential equation. What are the assumptions necessary for this equation to be correct?
(b) What can you say about the relation between the electric and magnetic field intensities under the given conditions?

11.19 Magnetic Vector Potential. Given: Maxwell's equations and the vector $\mathbf{B} = \nabla \times \mathbf{A}$, in a linear, isotropic, homogeneous medium. Assume that $\mathbf{E} = 0$ for static fields:

(a) By neglecting the displacement currents, show that Maxwell's equations reduce to a second-order partial differential equation in $\mathbf{A}$ alone.
(b) What is the electric field intensity?
(c) Show that by using the Coulomb's gauge, the equation in **(a)** is a simple Poisson equation.

11.20 An Electric Vector Potential. A vector potential may be derived as $\nabla \times \mathbf{F} = -\mathbf{D}$ where **D** is the electric flux density:

(a) What must be the static magnetic field intensity (other than $\mathbf{H} = 0$ or $\mathbf{H} = \mathbf{C}$, where **C** is a constant vector) if we know that in the static case, $\nabla \times \mathbf{H} = 0$?
(b) Find a representation of Maxwell's equations in terms of the vector potential **F** in a current-free region (i.e., a region without source currents).
(c) What might the divergence of **F** be for the representation in **(b)** to be useful? Explain.

11.21 Modified Magnetic Vector Potential. A modified vector potential may be defined as $\mathbf{F} = \mathbf{A} + \nabla\psi$, where **A** is the magnetic vector potential as defined in **Eq. (11.40)** and ψ is any scalar function:

(a) Show that this is a correct definition of the vector potential.
(b) Find an expression of Maxwell's equations in terms of **F** alone.
(c) How would you name the two potentials **F** and ψ?

Interface Conditions for General Fields

11.22 Displacement Current Density in a Dielectric. A time-dependent electric field intensity is applied on a dielectric as shown in **Figure 11.7**. The electric field intensity in free space is given as $\mathbf{E} = \hat{\mathbf{z}}E_0\cos\omega t \quad [\mathrm{V/m}]$. The relative permittivity of the material is $\varepsilon_r = 25$. For $E_0 = 100$ V/m and $\omega = 10^9$ rad/s, calculate the peak displacement current density in the dielectric (there are no surface charges at the interface between air and material).

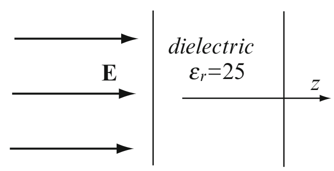

Figure 11.7

11.23 The Hertz Potential. In a linear, isotropic, homogeneous medium devoid of sources, one can derive the fields from a single potential called the Hertz potential, $\boldsymbol{\pi}$, as follows:

$$\mathbf{A} = j\omega\mu\varepsilon\boldsymbol{\pi}, \qquad V = -\nabla\cdot\boldsymbol{\pi}$$

where **A** is the magnetic vector potential and V the electric scalar potential.
Find the expressions for the electric and magnetic field intensities to show that they are dependent on $\boldsymbol{\pi}$ alone.

11.24 The Use of a Gauge. In a linear, isotropic, homogeneous medium devoid of sources, one can define the magnetic Hertz potential $\boldsymbol{\pi}_m$ as

$$\mathbf{E} = -j\omega\mu\nabla\times\boldsymbol{\pi}_m \quad [\mathrm{V/m}]$$

Show that one can write Maxwell's equations in the frequency domain in terms of $\boldsymbol{\pi}_m$ alone provided a proper gauge is defined. What is that gauge?

11.25 Interface Conditions for General Materials. Two dielectrics meet at an interface (see **Figure 11.8**) at $x = 0$. A sinusoidal electric field intensity of peak value 5 V/m and frequency 1 kHz exists in dielectric (1). For $x < 0$, $\varepsilon = 2\varepsilon_0$ [F/m], and $\mu = \mu_0$ [H/m]. For $x > 0$, $\varepsilon = 3\varepsilon_0$, and $\mu = 2\mu_0$. If the electric field intensity vector is incident at 30 from the normal, find the magnitudes of **E** and **D** on each side of the interface. Assume no current or charge densities exist at the interface.

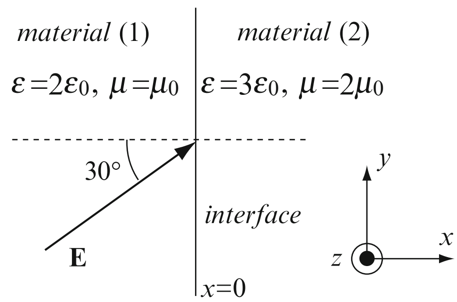

Figure 11.8

11.26 Calculation of Fields Across Interfaces. A region, denoted as region (1), occupies the space $x < 0$ and has relative permeability $\mu_{r1} = 6$. The magnetic field intensity in region (1) is $\mathbf{H}_1 = \hat{\mathbf{x}}4 + \hat{\mathbf{y}} - \hat{\mathbf{z}}2$ [A/m]. Region (2) is defined as $x > 0$ with $\mu_{r2} = 5.0$. No current exists at the interface. Find **B** in region (2).

11.27 Interface Conditions for Permeable Materials. An interface between free space and a perfectly permeable material exists. In free space (1), $\mu = \mu_0$ [H/m], $\varepsilon = \varepsilon_0$ [F/m], and $\sigma = 0$. In the permeable material (2), $\mu = \infty$, $\sigma = 0$, and $\varepsilon = \varepsilon_0$. Define the interface conditions at the interface between the two materials.

11.28 Surface Current Density at Interfaces. Two magnetic materials meet at an interface as shown in **Figure 11.9**. Material (1) has relative permeability of 4 and material (2) has relative permeability of 2. The interface is at $z = 0$. The magnetic flux density in material (1) is given as $\mathbf{B} = \hat{\mathbf{x}}0.1 + \hat{\mathbf{y}}0.2 + \hat{\mathbf{z}}0.1$ [T]. In material (2), it is known that all tangential components of **H** are zero.

(a) Calculate the surface current density that must exist on the interface for this condition to be satisfied.
(b) Calculate the magnetic flux density in material (2).

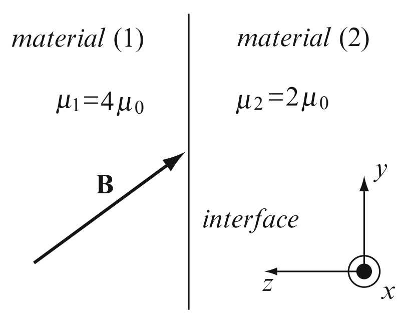

Figure 11.9

11.29 Simulated Surface Current Density. It is possible to simulate a current sheet at an interface by placing thin parallel wires at the interface. Suppose two materials meet at an interface on the x–y plane at $z = 0$. Both materials are the same, with relative permeability $\mu_r = 2$. The magnetic field intensity in material (1) ($z > 0$) is given as $\mathbf{H}_1 = \hat{\mathbf{x}}10^5 + \hat{\mathbf{y}}2 \times 10^5 + \hat{\mathbf{z}}10^4$ [A/m]. Suppose now that wires are placed on the interface such that the current in the wire points at 45° to the x axis, as shown in **Figure 11.10**. The current in each wire is 0.1 A and there are two wires per mm length. Calculate:

(a) The magnetic field intensity in material (1) and in material (2) before the current in the wires is added.
(b) The magnetic field intensity in both materials after the current is switched on.

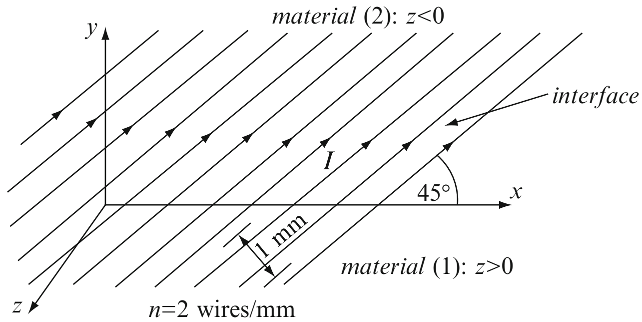

Figure 11.10

11.30 Simulated Surface Current Density. Suppose that in the previous problem, the magnetic field intensity in material (1) with the same current sheet (i.e., the total field in material (1) due to all sources, including the current sheet) is given as $\mathbf{H}_1 = \hat{\mathbf{x}}10^5 + \hat{\mathbf{y}}2 \times 10^5 + \hat{\mathbf{z}}10^4$ [A/m]:

(a) What is now the magnetic field intensity in material (2).
(b) Discuss the difference between the solution to this problem and the previous problem.

Time-Harmonic Equations/Phasors

11.31 Vector Operations on Phasors. Two complex vectors are given as $\mathbf{A} = \mathbf{a} + j\mathbf{b}$ and $\mathbf{B} = \mathbf{c} + j\mathbf{d}$, where **a**, **b**, **c**, and **d** are real vectors. Calculate (* indicates complex conjugate):

$$\mathbf{A}\cdot\mathbf{A} \quad \mathbf{A}\cdot\mathbf{A}^* \quad \mathbf{A}\cdot\mathbf{B} \quad \mathbf{A}\cdot\mathbf{B}^*$$
$$\mathbf{A}\times\mathbf{A} \quad \mathbf{A}\times\mathbf{A}^* \quad \mathbf{A}\times\mathbf{B} \quad \mathbf{A}\times\mathbf{B}^*$$

11.32 Conversion of Phasors to the Time Domain. A magnetic field intensity is given as $\mathbf{H} = \hat{\mathbf{y}}5e^{-j\beta z}$ [A/m]. Write the time-dependent magnetic field intensity.

11.33 Conversion to Phasors. The following magnetic field intensity is given in a domain $0 \le x \le a$, $0 \le y \le b$:

$$H(x, y, z, t) = H_0 \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \cos(\omega t - kz) \quad [\text{A/m}]$$

where x, y, and z are the space variables, m and n are integers, and k is a constant. Find the rectangular, polar, and exponential phasor representations of the field.

11.34 Conversion to Phasors. An electric field intensity is given as

$$E(z, t) = E_1 \cos(\omega t - kz + \psi) + E_2 \cos(\omega t + kz + \psi) \quad [\text{V/m}]$$

Write the phasor form of E in polar and exponential forms.

11.35 Conversion of Phasors to the Time Domain. A phasor is given as

$$E(x, z) = E_0 e^{-j\beta_0(x\sin\theta_i + z\cos\theta_i)} \quad [\text{V/m}]$$

where x and z are variables and β_0 and θ_i are constants. Find the time-dependent form of the field E.

11.36 Conversion to Phasors. The electric field intensity in a domain is given as

$$E_x(z, t) = E_0 \cos(\omega t - kz + \phi) \quad [\text{V/m}]$$

Find:

(a) The phasor representation of the field in exponential form.
(b) The first-order time derivative of the phasor.

11.37 Time-Harmonic Fields. The electric field intensity

$$\mathbf{E} = \hat{\mathbf{x}}10\pi\cos\left(10^6 t - 50z\right) + \hat{\mathbf{y}}10\pi\cos\left(10^6 t - 50z\right) \quad [\mathrm{V/m}]$$

is given in a linear, isotropic, homogeneous medium of permeability μ_0 [H/m] and permittivity ε_0 [F/m]. Write the magnetic field intensity and the magnetic flux density:

(a) In terms of the time-dependent electric field intensity.
(b) In terms of the time-harmonic electric field intensity.

11.38 Time-Harmonic Fields. The magnetic field intensity in free space is given as $\mathbf{H} = \left(\hat{\mathbf{x}}H_x + \hat{\mathbf{y}}H_y + \hat{\mathbf{z}}H_z\right)e^{j\beta z}e^{j\phi}$ [A/m], where H_x, H_y and H_z are complex numbers given as $H_x = h_x + jg_x$, $H_y = h_y + jg_y$ and $H_z = h_z + jg_z$:

(a) What is the time-dependent magnetic field intensity **H** in air?
(b) Write the magnetic field intensity in terms of amplitude and phase.

11.39 Two vector fields are given in phasor form as

$$\mathbf{E}_1 = \hat{\mathbf{x}}(20 + j20)e^{j0.3\pi z} + \hat{\mathbf{y}}(10 - j20)e^{j0.3\pi z}, \quad \mathbf{E}_2 = -\hat{\mathbf{x}}(20 - j10)e^{j0.3\pi z} + \hat{\mathbf{y}}(20 + j20)e^{j0.3\pi z} \quad [\mathrm{V/m}]$$

Calculate:

(a) The time domain representation of the two fields.
(b) The sum $\mathbf{E}_1 + \mathbf{E}_2$ in phasor form and in the time domain.
(c) The difference $\mathbf{E}_1 - \mathbf{E}_2$ in phasor form and in the time domain.
(d) The vector product of the two fields in the time domain and in phasor form.
(e) The scalar product of the two fields in the time domain and in phasor form.