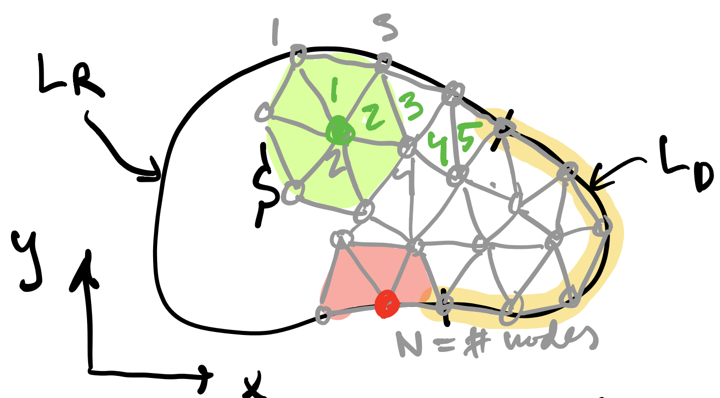
FEM in 2D for scalar Helmholtz' eq.

$$\begin{cases} -\nabla\cdot(\alpha\nabla f)+\beta f = s & \text{in } S \\ f = p & \text{on } L_D \\ \hat{n}\cdot(\alpha\nabla f)+\gamma f = q & \text{on } L_R \end{cases}$$



D = Dirichlet

R = Robin

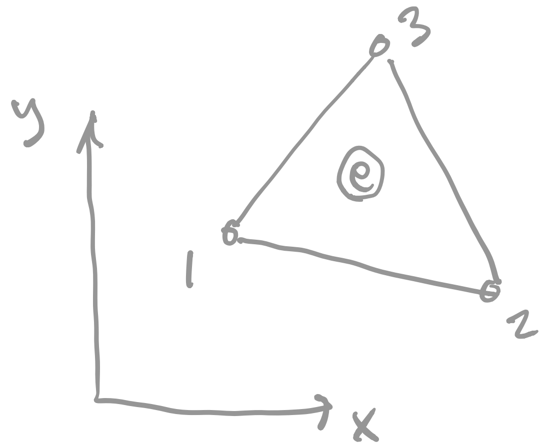
(N = Neumann ⇔ Robin BC with $\gamma = 0$)

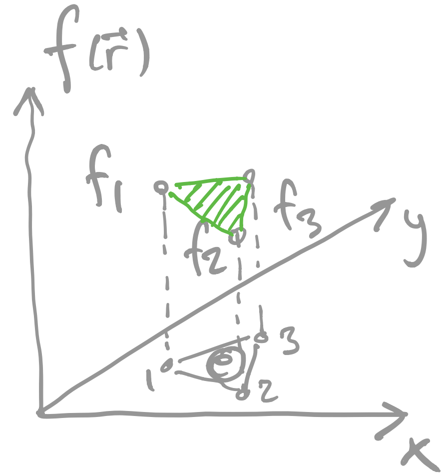
$$f = f(\vec{r}) = f(x,y) = \sum_{j=1}^{N} f_j \varphi_j(\vec{r})$$

$$\varphi_j(\vec{r}_i) = \begin{cases} 1 & \text{at } i=j \\ 0 & \text{at } i\neq j \end{cases}$$

$\varphi_j(\vec{r})$ are piecewise linear functions of x and y

[$\varphi_j(\vec{r})$ is a linear function of x & y inside each triangle]





For element ⓔ, we have

$$\varphi_1^{(e)}(x, y) = a^{(e)} + b^{(e)} x + c^{(e)} y$$

$$\begin{cases} \varphi_1^{(e)}(x_1^{(e)}, y_1^{(e)}) = a_1^{(e)} + b_1^{(e)} x_1^{(e)} + c_1^{(e)} y_1^{(e)} = 1 \\ \varphi_1^{(e)}(x_2^{(e)}, y_2^{(e)}) = a_1^{(e)} + b_1^{(e)} x_2^{(e)} + c_1^{(e)} y_2^{(e)} = 0 \\ \varphi_1^{(e)}(x_3^{(e)}, y_3^{(e)}) = a_1^{(e)} + b_1^{(e)} x_3^{(e)} + c_1^{(e)} y_3^{(e)} = 0 \end{cases}$$

$$\Rightarrow \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a_1^{(e)} \\ b_1^{(e)} \\ c_1^{(e)} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

We have the diff. eq. $\mathcal{L}(f) = s$ with $\mathcal{L} = -\nabla\cdot(\alpha\nabla f) + \beta f$, which gives the residual

$$r = \mathcal{L}(f) - s = -\nabla\cdot(\alpha\nabla f) + \beta f - s$$

Choose a weighting functions $w_i = \varphi_i$ (for $i = 1, 2, \ldots, N$) which gives

$$\int_{\xi} w_i\, r\, d\xi = 0 \quad \text{for } i = 1, 2, \ldots, N$$

(except i = node with Dirichlet boundary cond.)

$$\Rightarrow \int_{\xi} w_i\left[-\nabla\cdot(\alpha\nabla f) + \beta f\right] d\xi = \int_{\xi} w_i\, s\, d\xi$$

Integrate by parts:

$$\nabla\cdot\left[w_i(\alpha\nabla f)\right] = \alpha\nabla w_i\cdot\nabla f + w_i\nabla\cdot(\alpha\nabla f)$$

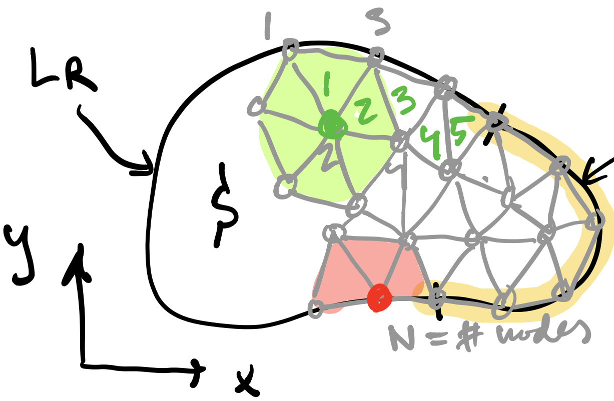
$$-w_i \nabla\cdot(\alpha\nabla f) = \alpha \nabla w_i \cdot \nabla f - \nabla\cdot\left[w_i(\alpha\nabla f)\right]$$

$$\int_S -w_i\nabla\cdot(\alpha\nabla f) + \beta w_i f \, dS = \int_S w_i s\, dS$$

$$\Rightarrow \int_S \alpha\nabla w_i\cdot\nabla f + \beta w_i f\, dS$$

$$- \oint_{L_D+L_R} \Big[\underbrace{w_i}_{=0 \text{ on } L_D}(\alpha\nabla f)\Big]\cdot\hat{n}\, dL =$$

$$= \int_S w_i s\, dS$$



L_D with $w_i = 0$

$$\int_{\Omega} \alpha \nabla w_i \cdot \nabla f + \beta w_i f \, d\Omega - \int_{L_R} \left[w_i (\alpha \nabla f) \right] \cdot \hat{n} \, dL = \int_{\Omega} w_i s \, d\Omega$$

Robin boundary condition

$$\hat{n} \cdot (\alpha \nabla f) + \gamma f = q \quad \text{on } L_R$$

$$\Rightarrow \hat{n} \cdot (\alpha \nabla f) = q - \gamma f$$

$$\int_{\Omega} \alpha \nabla w_i \cdot \nabla f + \beta w_i f \, d\Omega - \int_{L_R} w_i (q - \gamma f) \, dL = \int_{\Omega} w_i s \, d\Omega$$

$$\int_S \alpha \nabla w_i \cdot \nabla f + \beta w_i f \, dS + \int_{L_R} w_i \gamma f \, dL =$$

$$= \int_S w_i s \, dS + \int_{L_R} w_i q \, dL$$

for $i = 1, 2, \ldots, N$
(except $i =$ node with Dirichlet boundary cond.)

Use $f(\vec{r}) = \sum_{j=1}^{N} f_j \varphi_j(\vec{r})$ and

$w_i(\vec{r}) = \varphi_i(\vec{r})$ to get

$$\sum_{j=1}^{N} f_j \Bigg[\overbrace{\int_{S} \alpha \nabla\varphi_i \cdot \nabla\varphi_j + \beta \varphi_i \varphi_j \, dS + \int_{L_R} \varphi_i \gamma \varphi_j \, dL}^{=A_{ij}} \Bigg] =$$

$$= \underbrace{\int_{S} \varphi_i s \, dS + \int_{L_R} \varphi_i q \, dL}_{b_i}$$

$$\Rightarrow \quad \sum_{j=1}^{N} A_{ij} f_j = b_i$$

for $i = 1, 2, \ldots, N$

(except i = node with Dirichlet boundary cond.)

$$\Rightarrow \mathbb{A}\mathbb{f} = \mathbb{b}$$

$$\Rightarrow \left[\begin{array}{c:c} \mathbb{A}_U & \mathbb{A}_K \end{array} \right] \begin{bmatrix} \mathbb{f}_U \\ \mathbb{f}_K \end{bmatrix} = \mathbb{b}$$

known coefficients given the Dirichlet boundary condition

$$\Rightarrow \mathbb{A}_U \mathbb{f}_U + \mathbb{A}_K \mathbb{f}_K = \mathbb{b}$$

$$\Rightarrow \mathbb{A}_U \mathbb{f}_U = \mathbb{b} - \mathbb{A}_K \mathbb{f}_K$$

$$\Rightarrow \mathbb{f}_U = \mathbb{A}_U^{-1} \left[\mathbb{b} - \mathbb{A}_K \mathbb{f}_K \right]$$