Convergence and extrapolation

Hand-in assignment # 1 – SSY200

1 Problem description

Consider the integral

$$\int_a^b f(x)dx \tag{1}$$

where $0 \leq a < b$. If $f(x)$ has a primitive function, the analytically calculated value of the integral (1) is denoted I_0.

2 Assignments

- Write a MATLAB-program that evaluates the integral (1) by means of midpoint integration, where the integrand is $f(x) = x^\xi$ for some given value of ξ. The program should allow for midpoint integration with n sub-intervals of length $h = (b-a)/n$. In the following, the value of the numerically evaluated integral is denoted by $I_{\text{midp}}(h)$.
- Execute your program for $f(x) = x^\xi$ with $a = 1$ and $b = 2$ for $\xi = -3/2, -1/2, 1/2, 3/2$.
  - Evaluate the absolute error $e(h) = |I_{\text{midp}}(h) - I_0|$ as a function of the resolution controlled by h, where you should use the expression for I_0 that you get from analytical integration. Does the computed result converge to the analytical answer? What's the order of convergence? Does the order of convergence agree with what you expect from an analysis of the problem? (Here, the wording "an analysis of the problem" implies that you use analytical calculations to determine the order of convergence, i.e. no usage of numerical computations.)
  - Extrapolate the numerically computed result to zero cell size: fit the function $I_{\text{model}}(h) = c_0 + c_\alpha h^\alpha$ to the computed data under the assumption that the constants c_0, c_α and α are unknown. How well does the model $I_{\text{model}}(h)$ compare with the computed data $I_{\text{midp}}(h)$? Compare the extrapolated value c_0 and the estimated order of convergence α with the analytical results. Test your extrapolation method on different sets of computed data. How does the data influence the possibilities for accurate extrapolation?
- Execute your program for $f(x) = x^\xi$ with $a = 0$ and $b = 2$ for $\xi = -3/2, -1/2, 1/2, 3/2$.
  - Evaluate the absolute error $e(h) = |I_{\text{midp}}(h) - I_0|$ as a function of the resolution controlled by h, where you should use the expression for I_0 that

you get from analytical integration. Does the computed result converge to the analytical answer? What's the order of convergence? Does the order of convergence agree with what you expect from an analysis of the problem?

  - Extrapolate the numerically computed result to zero cell size: fit the function $I_{\text{model}}(h) = c_0 + c_\alpha h^\alpha$ to the computed data under the assumption that the constants c_0, c_α and α are unknown. How well does the model $I_{\text{model}}(h)$ compare with the computed data $I_{\text{midp}}(h)$? Compare the extrapolated value c_0 and the estimated order of convergence α with the analytical results. Test your extrapolation method on different sets of computed data. How does the data influence the possibilities for accurate extrapolation?

- Update and execute your program for the new integrand $f(x) = \sin(x^2)$ with $a = 0$ and $b = 100$.

  - Evaluate the integral with your program that implements the midpoint integration rule and use 2^n subintervals, where $n = 0, 1, 2, \ldots, 20$. Plot the integrand first. How many subintervals are necessary to reach the asymptotic region of convergence? What's the extrapolated answer to the integral? What's the order of convergence? Do these results agree with your expectations?

3 Report

Compare and explain your findings in the report. It is important that you try to provide mathematical arguments to support your conclusions. Do not forget to

- describe your midpoint integration scheme by means of the `MATLAB`-program and suitable derivations,
- describe your extrapolation scheme by means of the `MATLAB`-program and suitable derivations, and
- show the derivations for the expected order of convergence and compare this with what the numerical simulation yield.